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Erasmus+ Programme
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Brochure

GEOMETRICIAN'S VIEWS

Explore the mathematical culture through arts and sciences

Project number: 2020-1-FR01-KA227-SCH-095534

2021-2023

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Project presentation

The Erasmus + project KA225-SCH-095534, Geometrician's views (GV) is an international cooperation in the field of education. The aim of the initiative is to enable pupils from kindergarten up to high school level to explore mathematical culture through art and science.

Partners from 7 different countries are invited to exchange innovative approaches towards teaching maths, students are invited to create collaborative deliverables at the intersection between mathematics and arts.

The initiative includes an international staff training open to the members of the participating organisations, as well as multiplier events to disseminate the project outcomes.

The GV consortium expects to produce a number of tangible outputs including supports for traveling exhibitions on maths and arts, guides for the teachers who wishing to implement hands-on activities on this topic in the classroom.

The strategic goals are:

- To disseminate project activities, findings and outcomes to stakeholders (NGOs, educational institutions, instructional environments, etc.).
- To identify and engage stakeholders throughout the project in order to widen the project stakeholder base.
- To facilitate and share understanding of educational approaches suitable for formal activities on maths and arts.
- To maximise the impact of the outcomes of the project (the educational resources, publications, etc.) .
- To raise awareness of innovation in education, especially in mathematics, among the general public.

In addition, this strategy comprises a more practical objectives:

- To motivate partners to communicate about the project on social media and online platforms.
- To prepare communication supports (flyers, posters, banners, etc.) and other materials for dissemination.

Website:
https://geometricians-views.lesmathsenscene.fr/

The partners

ITALY: Istituto Comprensivo Perugia 14

SPAIN: Virtual Dor

SERBIA: Petro Kuzmjak school

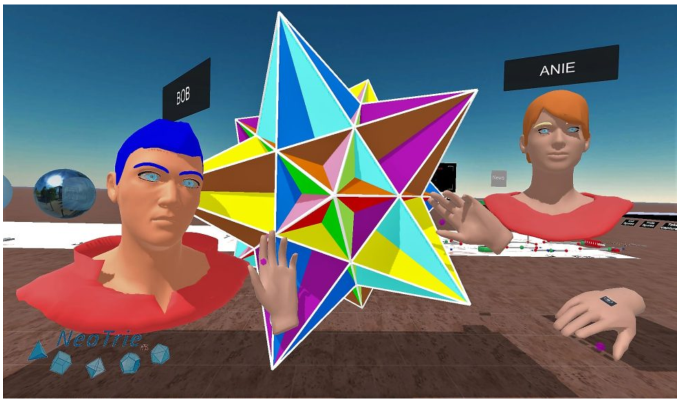
ROMANIA: Middle school Ioan Ciordas

CROATIA: High school Novska

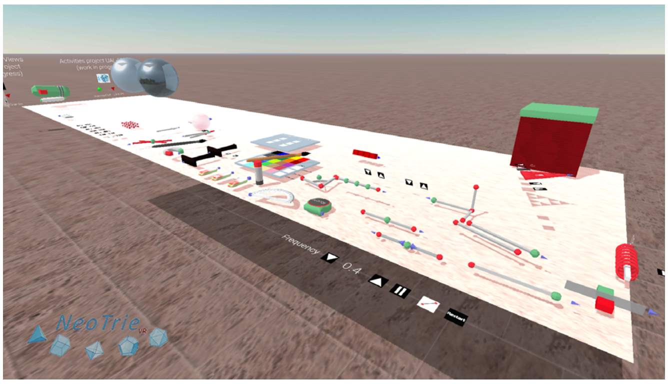
GREECE: Geniko Lykeio Proastiou Karditsas

FRANCE: L'association Les Maths En Scène

Neotrie VR

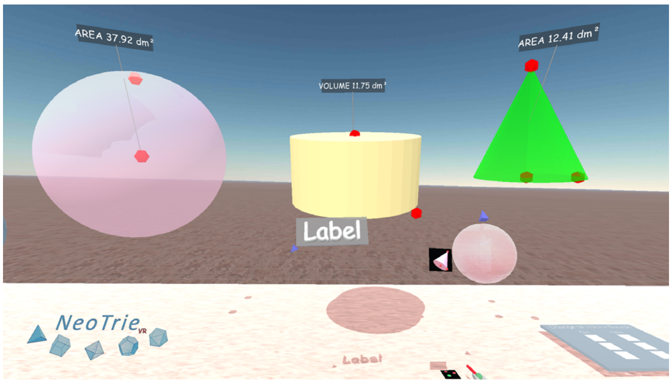


The standard tools of dynamic 3D geometry can be operated intuitively in the VR environment:



NeoTrie VR is a multiplayer virtual reality software, that allows users to create, manipulate and interact with geometric objects and 3D models in general, of different types, in a collaborative way.

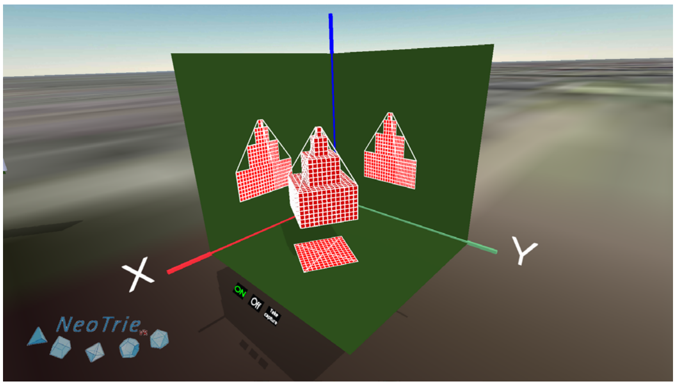
- 2D and 3D geometry (parallels, perpendiculars, bisectors,...) and metric calculations (lengths, angles, areas, volume,...).



The aims of the use of this software are:

- Plane geometry visible through the eyes of a third dimension;
- 3D geometry and modelling meant for 3D printing;
- Develops handicrafts and 3D visual skills;
- Stimulates deductive and inductive reasoning skills;
- Fosters cooperative work and positive interdependence;
- Motivates pupils by means of recreational, collaborative and competitive games.

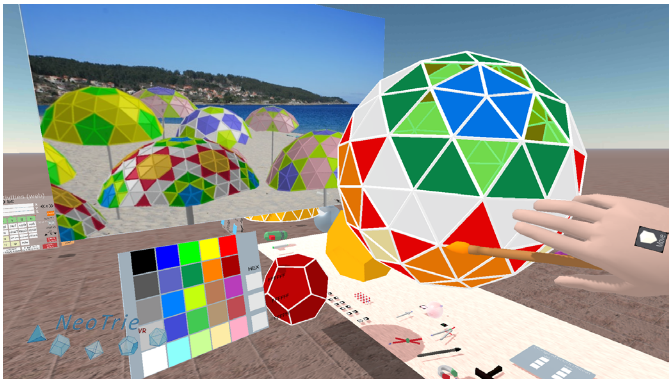
- Projection of 3D figures in multiview planes.



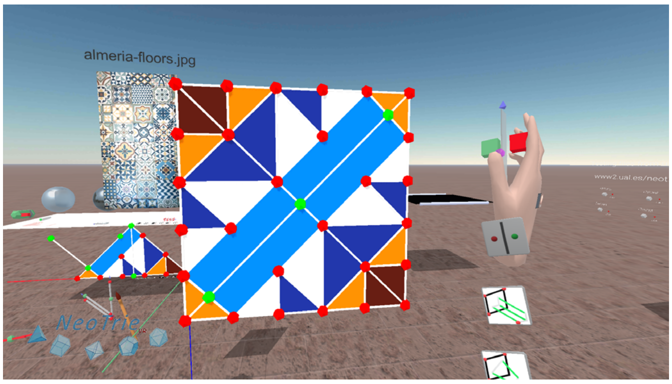
The software implements learning activities for different ages, covering a wide variety of topics.

Geometricians View's

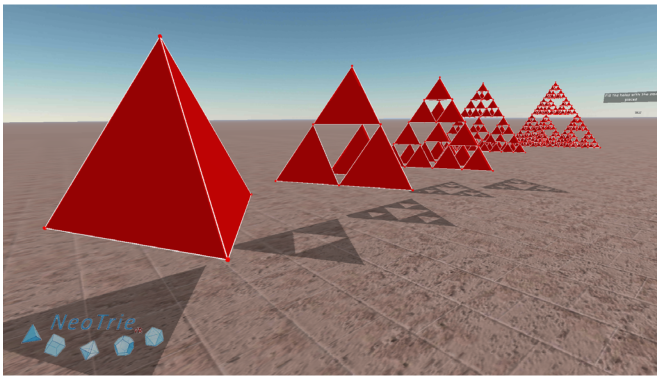
- Construction of famous polyhedra (Platonic bodies, pyramids, prisms, antiprisms, Archimedean, Johnson, Kepler, geodesic spheres, etc.), duality, truncation, rectification, extension,....



- Spatial symmetries to build 2d and 3d tessellations (ranslations, rotations, reflections,...).

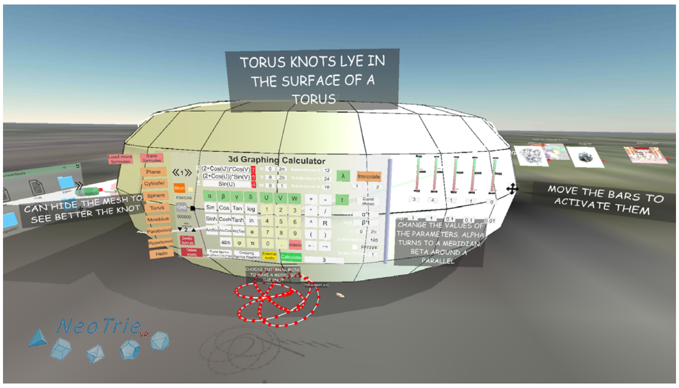


- Homotheties to build self similar fractals.

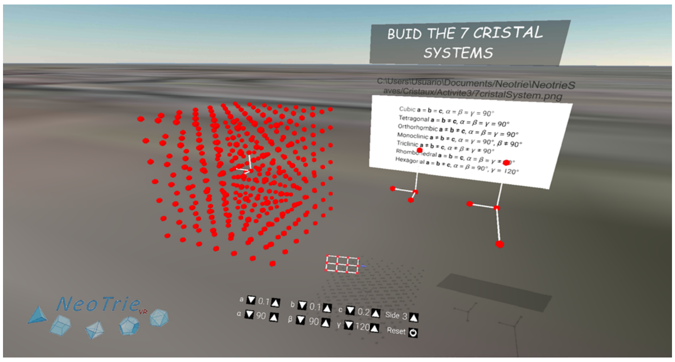


- Manipulate curved bodies, find intersecting curves.

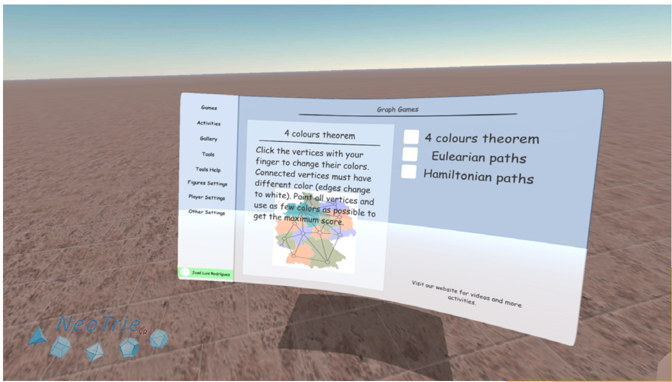
- 3d graphing calculator for parametrized curves and surfaces.



- Crystal networks from fundamental cells.



- Eulerian paths, Hamiltonian paths on graphs, colouring graphs.

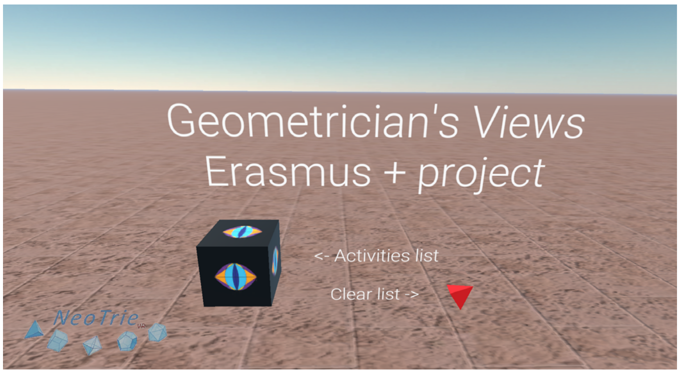


Features of Neotrie VR:

- Gallery of pre-designed geometric figures.
- File system to load and save activities created by users online.
- Photo camera inside the VR scene.
- Photos, videos, texts, sounds can be inserted in the scene.
- Exporting-importing from other 3D geometric softwares as STL objects.
- Importing GeoGebra files (alpha version).
- 3D printing from STL files edited in NeoTrie.
- Multiplatform: compatible with Meta, Steam VR, Vive, Valve Index, Pico4, etc.

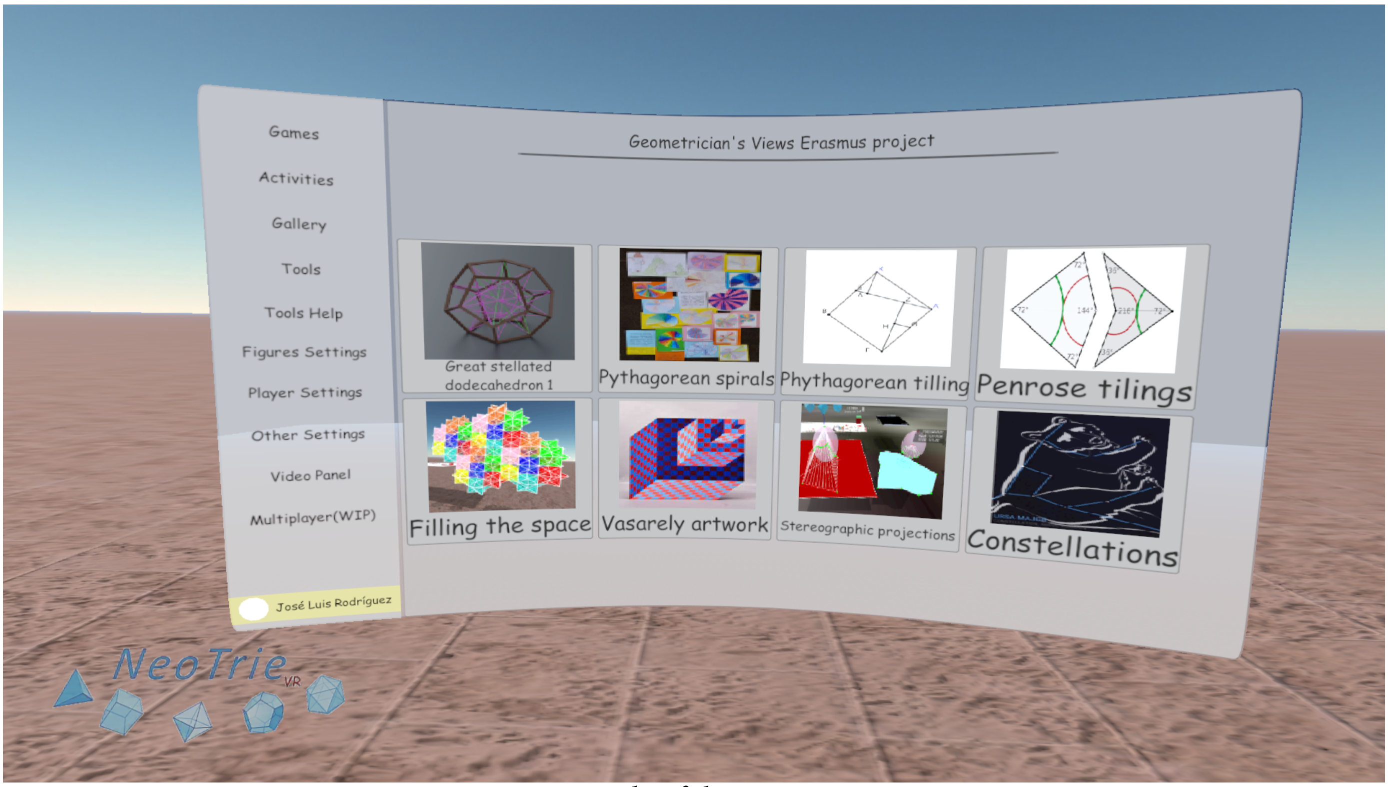
Interactiv Virtual Museum

In this brochure you will find extra activities in Neotrie VR, which are accessible in the game by touching this cube in the scene.



Visit the following web page for detailed information on how to implement Neotrie VR in the classroom, and the complete list of activities:

https://www2.ual.es/neotrie/project/geomview/



A sample of the activities

Tutorials

Anamorphoses

Mathematics, Nature, Science, Art = Beauty

Anamorphosis has been used by many painters, like Felix Labisse, Savaltor Dali, Holbein.

Today, it has multiple applications both in the field of architecture and trompe-l'oeil . It is part of our daily life through traffic signs (bike path, pedestrian crossing with embossed patterns), advertisements on soccer fields, etc.

Anamorphosis is a reversible deformation of an image trough an optical system such as a curved mirror or a mathematical process.

The word ananamorphosis comes from the Greek **ανα μορφωειν** anamorphoein which means "to transform". Through this process, some artists have created distorted works which take shape from a pre-established point of view. In the past, anamorphosis was one of the applications in Piero della Francesca's work on perspective (1492).

Educational goal

Discover anamorphosis

Target Age

From 9 years old

Skill required

Using coordinates on a grid, mastering precise lines and measurements

Mathematical and artistic goal

Understand some techniques to obtain anamorphosis; put them into practice, invent, create

Learning activities

Activity 1: Discovery of anamorphosis

Introduce different anamorphosis from different angles, in order to have the students understand that it is a deformation and that the point of view of the viewer is essential.

Age: from 7/8 years old

Working mode: in large groups

Equipment: a video projector, pictures on paper sheets

Teacher worksheet: https://drive.google.com/file/d/108MVY2CZpTmozU0Wqy9n2_rOL_KG7sy5/view?usp=sharing

Activity 2: Creating a collective anamorphosis

In the playground, create an anamorphosis that can be seen by everyone. Cooperate to achieve the expected result. Concept of scale and proportionality.

Age: from 9/10 years old

Working mode: in half group

Equipment: floor chalk, large ruler and blackboard square, rope

Teacher worksheet: https://drive.google.com/file/d/1XQWFwkiwWzRNWi9rakrZO-n0oHz_RrnsC/view?usp=sharing

Activity 3: Creating an Individual Anamorphosis (the Cube)

Realization of an anamorphosis of a cube, on a paper sheet.

Age: from 9/10 years old

Working mode: in half group or smaller group (to guide the students as well as possible)

Equipment: square (or compass for older children), ruler

Teacher worksheet: https://drive.google.com/file/d/1IZaJotH0fIh-Eg9Cxiye6gE3arncr_YD/view?usp=sharing

Activity 4: Presentation on anamorphosis

Age of students: from 9 years old

Working mode: in groups

Equipment: Computer if slideshow or poster

Aim: working on oral skills

Teacher's worksheet: https://drive.google.com/file/d/1ogF-B_g0IxwfebfgK5VvwIz6jPM-sE3Rv/view?usp=sharing

Student work: https://drive.google.com/file/d/1z9LrwiDJ4tS7QRxWzgJI-O8RKSZB6D-mq/view?usp=share_link

Activity 5: Using dynamic geometry and anamorphosis software

Age of students: from 14 years old

Working mode: individual

Equipment: computer with Geogebra software

Aim: to strengthen and improve knowledge on geometric configurations.

Teacher's worksheets: https://drive.google.com/file/d/1d0MPykioWC5fnyQ3jfy7LOOj-8ZIXAHKP/view?usp=sharing

Student work: https://drive.google.com/file/d/1mzr42P3bITqyQHGeL8602N9Ws6ZQ-zUs6/view?usp=sharing

Activity 6: Conical Anamorphosis

Age of students: from 11 years old

Working mode: individual

Equipment: paper sheet, cylindrical mirror (built with ice cream cone and mirror sheet) + transfer ruler, spreadsheet

Aim: to create a conical anamorphosis

Teacher's worksheet: https://docs.google.com/document/d/1dky3bMDV0FRMD9Wh-H7jlQ1K2RYB1Og2uyCujgowM2VY/edit?usp=sharing

Student work: https://docs.google.com/document/d/1ShtZz9K7Uj1xx7rZMO0VZj86e-qQcv5OViDOHe6lPSjI/edit?usp=sharing

Activity 7: Cylindrical anamorphosis

Cylindrical anamorphosis is the distorted image of an object trough optical systems such as curved mirrors.

Age of students: from 9 years old

Working mode: individual

Equipment: paper sheet, square-divided grid, circular grid and cylindrical mirror(made with toilet paper roll and mirror sheet)

Aim: create a cylindrical anamorphosis

Teacher's worksheet: https://drive.google.com/file/d/1OVck1fwyERMnib53_vt2mke-J3kmSFA7D/view?usp=sharing

Student work: https://drive.google.com/file/d/1ml8V1PWsBp08IJA9QWdJ8ounUFoQka-RS/view?usp=sharing

Activity 8: Anamorphosis in middle school

The project consists of making anamorphoses in different parts of the school, materialised with colored duct tape. These anamorphoses are made from the students' drawings and then projected using devices designed for the occasion.

Working skills: multi-representation, understanding of alignment, proportionality (scales, similar triangles, dilation)

Age of students: 11 to 13 years old

Working method: first in small groups, then in large groups.

Equipment: colored masking tape, sheets of brown paper, marker pens, devices designed to project a drawing from a transparency film onto school surfaces, tarp, string, geometry materials.

Aim: to create permanent anamorphoses in the school, including at least one large one. Resources to improve students' culture: Boucq, Rousse, Varini, Zinn, Superliminal, Ok go, Museum of illusion, example in a middle school near Versailles, Abelanet.

Extra activity in Neotrie VR

Build anamorphoses of a cube or another figure in virtual reality.
https://www2.ual.es/neotrie/project/geomview/#Anamorphoses

Documents

Geogebra applet:
https://www.geogebra.org/m/6050

Two helpful documents:
https://maths.ac-noumea.nc/IMG/pdf/Expe_rience_en_Sixie_me_et_Cinquie_me.pdf

Laws of optics in Activity 6:
https://docplayer.fr/18223712-Les-anamorphoses-atelier-scientifique-college-antony-duvivier-3-rue-des-canes-58170-luzy.html

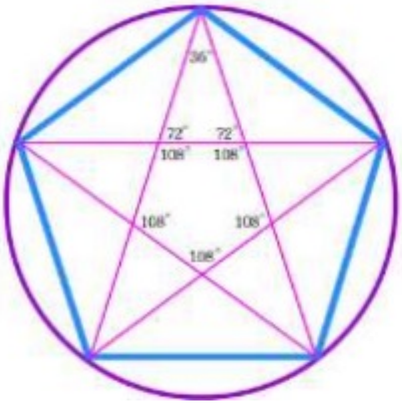
Webography

An activity on anamorphosis: Un EPI sur l'anamorphose - Mathématiques - Pédagogie - Académie de Poitiers (ac-poitiers.fr)
https://methodeheuristique.com/les/anamorphoses/
https://creapills.com/georges-rousse-anamorphoses-murs-illusion-20200304
https://maths.ac-noumea.nc/IMG/pdf/Expe_rience_en_Sixie_me_et_Cinquie_me.pdf
https://mutuamath.sesamath.net/taxonomy/term/2473
https://www.hisour.com/fr/anamorphosis-17938/

An anamorphosis software:
https://www.anamorphosis.com/software.html

Measurements and plots

Mathematics, Nature, Science, Art = Beauty



How have instruments and units of measurement evolved throughout history?

The Egyptians already used the 13-knot rope for construction. In the Middle Ages, builders did not have our precise technologies at their disposal to undertake defensive and religious building work. The measuring instruments were simple to use, not very cumbersome and above all could be carried everywhere. They were easy to understand by all, even by workers who could neither read nor write... In the Middle Ages, people counted on their fingers and used their body parts. So from this practice, units of measurement with explicit names were derived: the thumb, the foot, the cubit,...

Nevertheless, these units differed from one individual to another. Builders also used the pigeon, the square, the archipendulum, the plumb line, the compass already used in Antiquity,... Their instruments have evolved and have come down to us in slightly different forms and have benefited from our advanced technologies.

We no longer speak of a 13-knot rope but of an electronic angle reader, the laser level has replaced the archipendulum and the plumb line, the laser rangefinder has replaced the pigeon. For tracings, the builders of the Middle Ages used primarily the natural elements at their disposal: charcoal, ochre, etc.

Educational goal

Discover ways of measuring and drawing through history

Target Age

15/18 years old

Skill required

Basic use of Python

Mathematical and artistic goal

Measure and plot with an algorithm and use it in geometrical art

Learning activities

Activity 1: Presentation on the different measuring and drawing instruments

Age: 15-18 years old

Working mode: in groups

Equipment: computer if slideshow or poster

Aim: to work on oral skills

Teacher worksheet:

Instructions	You are going to make group presen-tations on the various measuring and plotting instruments. You will do your research in free time then an oral 3 to 5 min presentation in groups (slide show or poster), in class.
Listening and evaluation of Student's presentations	Possibility of using a co-evaluation grid with the other students

Student worksheet:

At home	Research on measuring and tracing ins-truments
Listening and evaluation of Student's presentations	Other students can participate in the as-sessment.

Activity 2: Turtle module

Age: 15-18 years old

Working mode: individual

Equipment: computer and Python (Edupython)

Aim: discover the Turtle module and the For loop

Teacher worksheet:

Instructions	This is an oral activity
Help during the activity	
Sharing on the activity	

Student work:
https://drive.google.com/file/d/1o7ylmL6L_89gaiSNRz6AmVXeBtx6Zz7X/view?usp=sharing

Activity in the classroom with EduPython	The Turtle module and the For loop
	1. What will the algorithm show?
	`from turtle import *`
	`for i in range(4):`
	`    forward(120)`
	`    left(90)`
	2. Change instructions to obtain a 100-sided square
	3. Change instructions to obtain a regular hexagon, an equilateral triangle and a regular octagon

Activity 3: Functions

Age: 15-18 years old

Working mode: individual

Equipment: computer and Python (Edupython)

Aim: to discover the functions

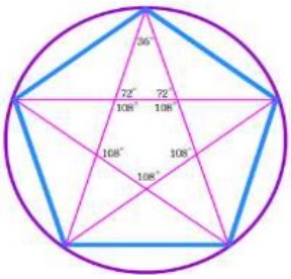
Teacher worksheet:

Instructions	This is an oral activity
Help during the activity	
Sharing on the activity	

Student work:

<table>
<tr>
<td>Activity in the classroom with EduPython</td>
<td>Introducing fonctions
1. What will the algorithm show?

```
from turtle import *
def triangle(longueur):
 for i in range(3):
     forward(longueur)
     left(120)
triangle(100)
```

2. Change instructions to obtain a 100-sided square
3. Change instructions to obtain a regular hexagon and a regular octagon
4. If you’re up to it, Change instructions to obtain a 5 point star.


</td>
</tr>
</table>

Activity 4: Making π-quills

General activity: using the series of works by François Morellet, π-piquant, π-rococo, etc., the pupils create geometric representations of the first decimal places of the number π, on various supports and by making choices of measurements.

Working skills: measuring an angle, drawing an angle of a given measurement, multi-representing, discovering the number π

Age of students: 11 to 13 years old

Equipment: supports (paper, cardboard, wood, sidewalk), floor chalk, rulers, protractors, compasses, rope, light string of fairy lights , etc.

Working mode: in large groups

Teacher worksheet: description of the work, progress of the activity, π-piquant in the city, link to come for the final realisations (an embroidery, a luminous realisation, a «city of maths» achievement, highlight of preparatory work)

Extra activity in Neotrie VR

Build the great stellated dodecahedron with 5 point stars.
https://www2.ual.es/neotrie/project/geomview/#Stellated

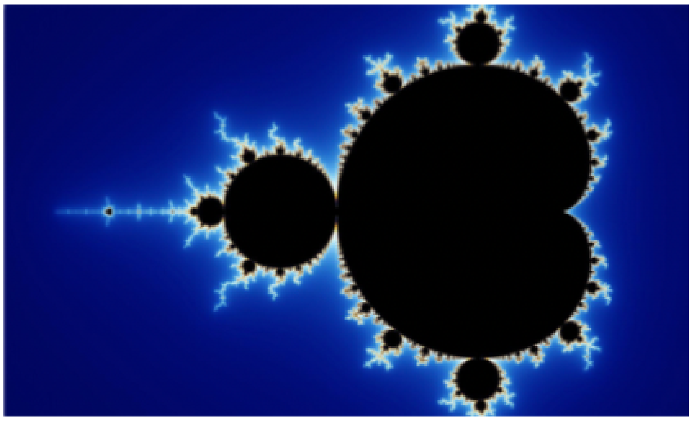
NeoTrie VR

Webography

https://youtu.be/VTIO3vhlhH0

Fractals

Mathematics, Nature, Science, Art = Beauty



Fractal
What is it?
A fractal is a geometric shape showing self-similarity. The same pattern is repeated infinitely at smaller scales.

The word "fractal" is from the Latin "fractus" which means "broken", "fractured".

Benoît Mandelbrot introduced fractal geometry in the seventies, when computers finally visualized the fascinating shapes produced by iterative mathematical procedures. These never-ending patterns were already well defined in his mind extremely gifted for geometric visualization. Fractal geometry proved to be an extraordinary approximation of many natural and scientific phenomena (trees, rivers, coastlines, mountains, clouds, seashells, hurricanes, galaxies).

Fractals are also one of the many artistic expressions of mathematics. The effect of a picture recursively appearing within itself, known as Droste effect, and the actual fractal art, a genre of computer and digital art, are examples of how the concept of self-similarity is also widespread and developed in art.

"Fractal geometry is not just a chapter of mathematics, but one that helps Everyman to see the same world differently" - Benoît Mandelbrot

Educational goal

Introduction to fractals as example of modern mathematics

Target Age

Pupils aged 12-14. Lower secondary school

Skill required

Basic knowledge of geometry, informatics and technical drawing

Mathematical and artistic goal

Example of a strong link between geometry and art

Activity 1: Examples of "iterative processes", "self-similarity" and "fractal generation rules"

After having studied and discussed these concepts, students will draw a recursive shape produced by the iterative application of a given rule. The mathematical rule can be proposed by students. They will discuss if the obtained drawing is self-similar.

Age: 12-13 years old

Equipment: ruler, graph paper, pencil, calculator, online notebook for using GeoGebra (https://www.geogebra.org/geometry).

Teacher worksheet:

Following these resources (Fractal Basics and Initiators and Generators), students learn the concepts of "initiator", "generator" and "rule" in order to draw their own "fractal" on graph papers. At the end of the activity, pupils exchange their drawings and identify the iterative process and verify the self-similarity obtained. Students try to use this online software, Recursive Drawing, to reproduce some of their drawings.

Student's work:

Italian students have performed again this activity in September 2021 in order to prepare the exhibition that was held on October 23rd, 2021. The same activity was done in December 2019.
«The FractalOne» - Progetto ErasmusPlus «Geometrician's Views» - 23 settembre 2021
"The FractalOne" - 23 settembre 2021

Activity 2: Examples of natural fractals

Students look for natural fractals discussing why the natural shapes selected can be described as fractals. They try to find the recursive rule by means of the online activity "Fractal Trees".

Age: 12-13 years old

Equipment: Pictures of natural fractals and for the ''Fractal Trees" activity: protractor, ruler with millimeters, calculator, fractal tree worksheet, pencil.

Teacher worksheet:

Students identify self-similarity in natural patterns (algae, flowers, trees, river networks, mountain ranges, snowflakes, proteins, crystals, Saturn rings, and so on); they draw on the picture printed out and looking for self-similarity. Within the "Fractal Trees" activity they measure the distance between the top of the tree (see worksheet) and different levels of branching, and also angles of branching. From distances ratios and angles, they are able to infer regular properties, trying to analyze patterns.

Student's work: Italian students carried out again this activity in September 2021 together with the "Fractal Trees" activity: Italian students working with natural fractals.

Students also learn about a recent work linking biology and fractals, summarized by the teacher: "Frattali: la geometria della realtà".

Activity 3

Students learn what Sierpinski and Mandelbrot fractals are. They use online resources to learn how GeoGebra and SCRATCH can be used to study and visualize fractals.

Age: 14 years old

Equipment: GeoGebra and SCRATCH online software for learning how to build a fractal by means of an iterative algorithm

Teacher worksheet:

This is an activity lead by an expert mathematician or physicist. Students first explore online examples of Sierpinski and Mandelbrot interactive fractals, then assisted by the expert code the algorithm needed to produce the fractal. They compare algorithms and fractals obtained.

Student's work:

Italian students started this coding activity in January 2020. We could not organize the afternoon meeting with the expert during this school year and we will do it in September 2021. Here a link to a tutorial prepared for that occasion. Students prepare their own tutorials explaining the procedure used.

Activity 4

Students explore the concept of fractal art, color fractal patterns printed from the web, and enjoy an online tool to visualize and produce different types of fractals.

Age of students: 12-14 years old

Equipment: Sierpinski and Mandelbrot fractals printed out and ready to color, colors, artworks printed out ready to analyze.

Teacher's worksheet:

This activity can be lead by an expert artist. Artworks to analyze and study with students: 1) Droste effect and Escher and the Droste effect, Hendrik Willem Lenstra: Escher and the Droste effect; The Droste Effect in Art; 2) The Face of War by Salvador Dali; 3) The great wave by Katsushika Hokusai. Using this online tool, students try to construct an artistic fractal.

Student work: Fractals colored by Italian students. Another activity was done in September 2021 and was part of the October exhibition.

http://www.manuelacasasoli.altervista.org/pagine/files_2021/frattali_regards.pdf
Exhibition Posters: October 23, 2021

Activity 5

Students learn what is the fractal dimension, read about the coastline paradox and chaos theory.

Age of students: 14 years old

Equipment: Online computers, a copybook, and a pen.

Teacher's worksheets: This activity was lead by an expert mathematician or physicist. Students learn how fractal geometry is currently important in science by a very simple introduction to the caos theory using the following resources: Fractal Dimensions - Koch Curve and Coastlines - What is Chaos Theory? - Chaos Theory.

Student work:

Discussion about the meaning of "Chaos Theory" - Simulations of simple and double pendulum (Caos per quattordicenni).

Extra activity in Neotrie VR
Build fractals in virtual reality with the fractal tool.
https://www2.ual.es/neotrie/project/geomview/#Fractals

NeoTrie VR

Documents

1. Maria Isabel Binimelis Bassa (2011) Un nuovo modo di vedere il mondo. I frattali. Mondo Matematico - RBA Italia S.r.l.

2. Benoît Mandelbrot (2005) Nel mondo dei frattali. Roma, Di Renzo.
Image 1: Mandelbrot Set - Image 2: Natural fractals

Webography

Learning activity 1: Recursive Drawing - Fractal Basics
Learning activity 2: Fractal Trees - Natural phenomena with fractal features (Wikipedia)
Learning activity 3: The Sierpinski Triangle - The Mandelbrot Set - GeoGebra - SCRATCH
Learning activity 4: Fractal Art - Online Fractal Tools
Learning activity 5: Fractal Dimensions - Koch Curve and Coastlines - What is Chaos Theory? - Chaos Theory

Pythagoras tree

Mathematics, Nature, Science, Art = Beauty

The Pythagorean tree is a fractal plane constructed from squares, invented by Professor Albert E. Bosman in 1942. It is a good example of modelling nature using fractal geometry.

The Pythagoras tree is a plane fractal constructed from squares. Its construction begins with a square. Upon this square are constructed two squares, each scaled down by a linear factor of sin(alpha), such that the corners of the squares coincide pairwise. The same procedure is then applied recursively to the two smaller squares, ad infinitum. The follow images show just 4 steps of the construction.

The Pythagoras tree is named after the Greek mathematician Pythagoras because each triple of squares encloses a right triangle. The Pythagoras tree is simply another form of proving the Pythagorean Theorem, which states that the square of the length of the hypotenuse of a right triangle equals the sum of the squares of each of the lengths of either two sides.

Educational goal

To study a fractal
To learn how geometrical properties remain while the new shape progressively become smaller.
To express the mathematical characteristics with the help of geogebra.
To use mathematics as a building block in constructions.

Target Age

Students and teachers

Skill required

Trigonometry (cos, sin), Geometry (Area, perimeter) Geogebra skills (create shapes, create a tool, using commands)

Mathematical and artistic goal

Students adopt a new motto, e.g. Let's Plant Pythagoras trees!

Students have the challenge to make impressive images.

Learning activities

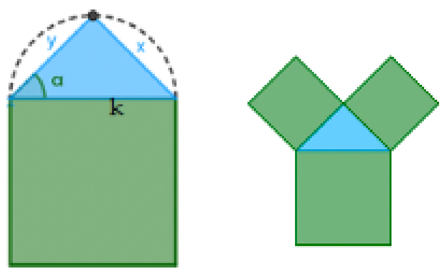
Activity

Age: 15-18 years old

Equipment: pencil, calculator, online notebook for using GeoGebra.

Teacher worksheet:

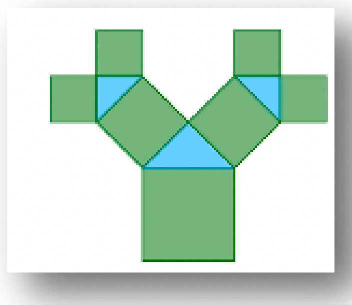
Steps of the construction-maths theory-questions for discussing



The initial shapes

The first square has side k. The blue right triangle has one of his angle a
The next two pairs of squares have the follow sides $x=k\sin\alpha$, $y=k\cos\alpha$, And for $\alpha=45^{\circ}$

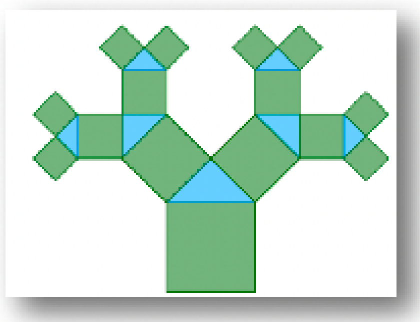
$$x=y=k\frac{\sqrt{2}}{2}$$



Step 2

The next four pairs of squares have the follow sides $x=k(\sin\alpha)^2$, $y=k(\cos\alpha)^2$
And for $\alpha=45^{\circ}$

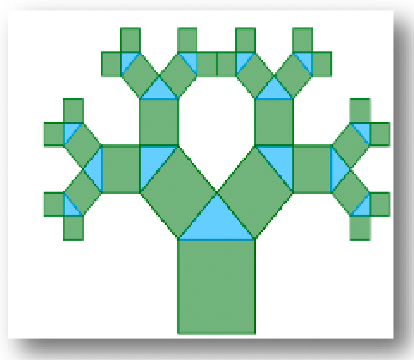
$$x=y=k\left(\frac{\sqrt{2}}{2}\right)^2$$



Step 3

The next eight pairs of squares have the follow sides
$x=k(\sin\alpha)^3$, $y=k(\cos\alpha)^3$
And for $\alpha=45^{\circ}$

$$x=y=k\left(\frac{\sqrt{2}}{2}\right)^3$$



step 4

The next 16 pairs of squares have the follow sides $x=k(\sin\alpha)^4$, $y=k(\cos\alpha)^4$
And for $\alpha=45^{\circ}$

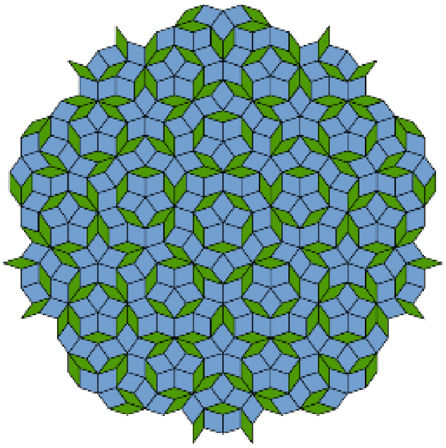
$$x=y=k\left(\frac{\sqrt{2}}{2}\right)^4$$

Extra activity in Neotrie VR

Play with a dynamic Pythagorean tree in virtual reality
https://www2.ual.es/neotrie/project/geomview/#Pythagorean tree

Penrose tilings

Mathematics, Nature, Science, Art = Beauty



Tiling - Tessellation
It deals with the covering of a plane surface or space with geometric shapes.
Tiling comes from the old English "tigele", from Latin "tegula", from an Indo-European root meaning "cover".

Tiling refers to a "collection of disjoint open sets, the closures of which cover the plane", whereas "a periodic tiling of the plane by polygons or space by polyhedra is called a tessellation" (Wolfram: Tiling and Tessellation). Tiling and tessellations are not only a prolific and challenging mathematical topic (Pentagon Tiling Proof Solves Century-Old Math Problem - The (Math) Problem With Pentagons - The trouble with five), but tessellation patterns are also common in nature, the most celebrated example being the hexagonal cells of honeycombs. Humans have copied these patterns in art, architecture, and technology, for a long time now.

The fertile production of Maurits Cornelis Escher is perhaps the most known example. "The regular division of the plane into congruent figures evoking an association in the observer with a familiar natural object is one of these hobbies or problems... I have embarked on this geometric problem again and again over the years, trying to throw light on different aspects each time. I cannot imagine what my life would be like if this problem had never occurred to me; one might say that I am head over heels in love with it, and I still don't know why."
Maurits Cornelis Escher

Educational goal	Target Age	Skill required	Mathematical and artistic goal
Covering surfaces with polygonal shapes is a valuable mathematical, artistic and educative activity	Pupils aged 13 - Lower secondary school	Basic knowledge of geometry and color theory	Recognizing mathematical patterns and encouraging artistic creativity

Learning activities

Activity 1

Age: 13 years old

Equipment: internet, GeoGebra, drawing papers, colors.

Teacher worksheet:

Students will learn the definition of “tiling” and “tessellation”, looking at natural, artistic, and mathematical examples. They will produce a tessellated surface with GeoGebra and reproduce the same one on drawing paper, combining colors as they prefer. They will also answer questions about which kind of polygons and why are more suitable for covering a surface.

Activity 2

Age: 13 years old

Equipment: Internet, GeoGebra, drawing papers, colors..

Teacher worksheet: Students will learn the definition of periodic and aperiodic tiling, studying the “Kite” and the “Dart” tiles. They will use the Penrose’s tiling to produce tessellated surfaces and discover the Phi number, the Golden Ratio, in their drawings.

Student’s work: Students’ work (Activities 1 and 2)

1. Student’s tutorial
2. Activities with the mathematician
3. Students’ work 1
4. Students’ work 2

Activity 3

Students will learn what Sierpinski and Mandelbrot fractals are. They will use online resources to learn how GeoGebra and SCRATCH can be used to study and visualize fractals.

Age: 13 years old

Equipment: Internet, drawing papers, colors.

Teacher worksheet: Students will study the artwork by Maurits Cornelis Escher. By doing translation, rotation, overturning, and reflection of tiles created by their own imagination, they will produce original artworks inspired by Escher’s work.

Student’s work: Students’ work (Activity 3)

1. Activity with the artist
2. Activity with the art teacher

Extra activity in Neotrie VR

Build the 7 Penrose vertex figures in virtual reality. Need to know the rules to join the pieces.
https://www2.ual.es/neotrie/project/geomview/#Penrose tilings

Documents

Tilings
Tiling - *Tessellation*
Tesselations
Image 1: Penrose Tiling - *Image 2: Butterfly M.C. Escher (1948)*

Webography

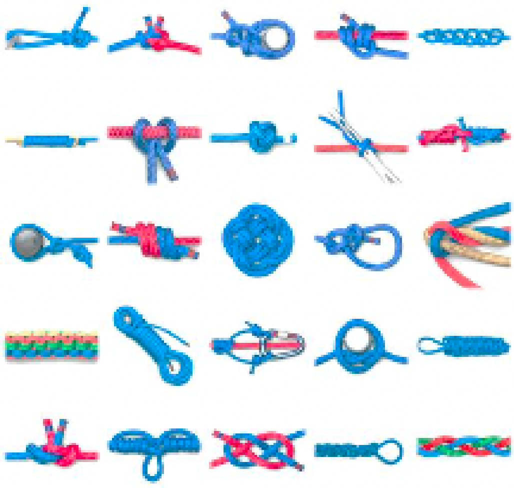
Learning activity 1:
Mathigon - Tessellation
What is a Tiling?

Learning activity 2:
Penrose Tilings and the Golden Ratio
What is... Penrose's Tiles?

Learning activity 3:
M.C. Escher Collection

Knots

Mathematics, Nature, Science, Art = Beauty



In mathematics, a knot is a simple curve enclosed in three-dimensional space. This mathematical object models a knot formed on a very thin rope whose ends have been joined.

In geometry, a knot is an immersion of a circle in three-dimensional Euclidean space (i.e. in space), considered to be continuous deformations. An essential difference between ordinary knots and mathematical knots is that the latter are closed (without heads that allow them to be tied or untied).

The physical properties of the knots themselves, such as the thickness of the strings, are usually neglected.

The branch of mathematics that studies knots is called knots theory. This theory has applications in physics (quantum field theory), chemistry and biology (the topology of DNA chains changes as a result of a biochemical process, favoring the formation of enzymes).

Educational goal

Students should know that knot theory is studied in mathematics, that the knot is a closed curve

Target Age

Between the ages of 12 and 18

Skill required

Basic knowledge of geometry, drawing and ICT

Mathematical and artistic goal

Students will identify the presence of mathematical knots in everyday life, they will have the opportunity to transfer knowledge and scientific notions in everyday life admiring what surrounds them and being creative in making the final product.

Learning activities

Activity 1: Theoretical notions about knots and presentation of examples

Age of students: 12-18 years

Material resources: computer, internet connection, string, scissors, tie, flipchart, YouTube documentaries

Activity details:
To the question "Where in real life do you encounter knots?

Students respond: Knots are a reality we are used to: a lace knot, a knotted hairstyle, a knitted sweater, a boat tied to shore.

Students are encouraged to follow the material at the address https://www.animatedknots.com/complete-knot-list and notice that the knots have imaginative names: sailor knot, Englishman's bow, boyfriend's knot, tie knot, pike's mouth, grandma's knot, etc. There is a mathematical theory of knots with applicability in molecular biology.

Each team will choose a knot pattern. The knot that gets the most votes will be made in the practical workshop.

Students read about the history of the «Gordian knot» and transpose it through art by making an essay or drawing, writing a poem or the lyrics of a song based on the story of the «Gordian knot».

Students are encouraged to follow the material: Famous patterns of tie knots https://lanieri.com/blog/fr/noeuds-de-cravate-tous-les-noeuds-les-plus-celebres-et-comment-les-faire/

Each student should choose a model and practice it at home. Take a picture of the tie with the knot and send it to the class group.

Activity 2: Making different knot models / Practical workshop

Age of students: 12-18 years

Material resources: computer, internet connection, string, scissors, flipchart, YouTube documentaries

Activity details: The students present the ties with the knot made by them. The most elegant and well-made tie knot is voted.

The following materials are followed:

Makrame knot art and basics (material can be translated into multiple languages) (https://www.youtube.com/watch?app=desktop&v=gWL3j5zLwhk) - Art of knots (material can be translated into multiple languages))

(https://www.youtube.com/watch?v=3QANzugFV0c)

We move on to the practical realization of the knots chosen in the previous activity, from the list: https://www.animatedknots.com/complete-knot-list

Extra activities in Neotrie VR

NeoTrie VR

A three-dimensional knot on the surface of a torus will be created, with the help of the 3d graphying calculator. Can change the parameters to change the type of the knot, and project these knots on the 3 multiview projection planes. A tutorial can be found at:
https://www2.ual.es/neotrie/project/geomview/#Torus knots

Also one can draw knots and links in virtual reality from the Rolfsen's table, with the free drawing options on our hands.
https://www2.ual.es/neotrie/project/geomview/#Knots

Bibliography / Resources:

https://delphipages.live/ro/stiinta/matematica/knot-theory

https://www.youtube.com/watch?v=3QANzugFV0c

https://www.youtube.com/watch?v=gWL3j5zLwhk

https://www.youtube.com/watch?v=-vms5MBOqq0&list=PLo_1gBBTiP1kJKgkuFGyqf92nARPiJ2EK

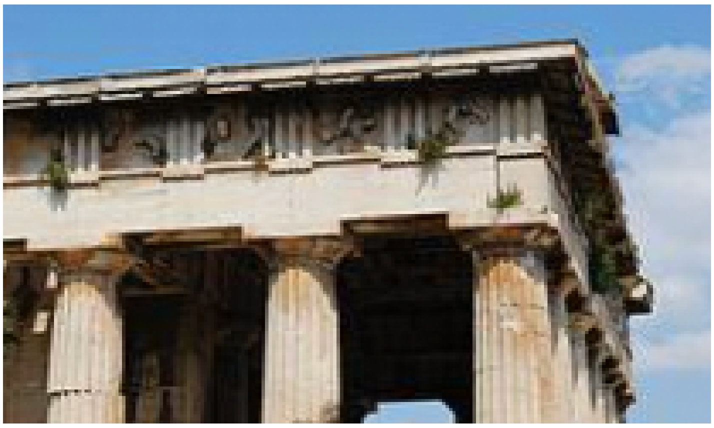
https://surequill.com/article/noeuds-de-cravate-tous-les-noeuds-les-plus-celebres-et-comment-les-faire

https://www.animatedknots.com/complete-knot-list

https://www.youtube.com/watch?v=co78AEqsv3s

Friezes

Mathematics, Nature, Science, Art = Beauty



A tessellation or tiling is the covering of a surface, often a plane, using one or more geometric shapes, called tiles, with no overlaps and no gaps. In mathematics, tessellation can be generalized to higher dimensions and a variety of geometries. A periodic tiling has a repeating pattern.

Mathematics and art go hand in hand throughout history. In the book «The Great Novel of Mathematics" - from prehistory to the present day", the author Mickaèl Launay argues that «in the mists of time there were many artists, researchers, inventors, creators, craftsmen or simply dreamers and curious who did mathematics without knowing. They were mathematicians, despite appearances! They were the first to ask questions, to research, to be the first to put their minds to work. "The author states that «If we want to understand the essence of mathematics, we must follow in their footsteps, because with them it all began.» https://www.ciordas.ro/erasmus/erasmus20/frize/Friezes%2C%20researcher%20view1.pdf

What are friezes? The friezes are those horizontal bands, presenting an identical motif that is repeated on the entire circumference of the vessel. They have different shapes. Passing from one epoch to another, it is observed that different motives appear. Some of them are repeated, others transformed or improved in various variants. If we analyse the frieze patterns, we find that they can be divided into seven categories, each associated with a group of different geometric transformations that leave the patterns unchanged. https://www.ciordas.ro/erasmus/erasmus20/frize/Friezes%2C%20artist%20view1.pdf

Educational goal

Students to know the notion of frieze in art, to discover the connection between the 7 types of friezes and geometric transformations (translations, rotations, symmetries)

Target Age

Students between 12 and 16 years old Lower secondary school

Skill required

Basic knowledge of geometry, drawing and ICT

Mathematical and artistic goal

Students will identify the presence of friezes in the environment, they will have the opportunity to transfer knowledge and scientific notions in everyday life admiring what surrounds them and being creative in making the final product.

Learning activities

Activity 1: Friezes in antiquity

Age: 12-16 years old

Equipment: computer, internet connection, coloured pencils, geometric tools, flipchart, YouTube documentation

Activity detail:

Theoretical notions synthesized from the book «The great novel of mathematics - from prehistory to the present day» by Micaël Launay are introduced to the students.

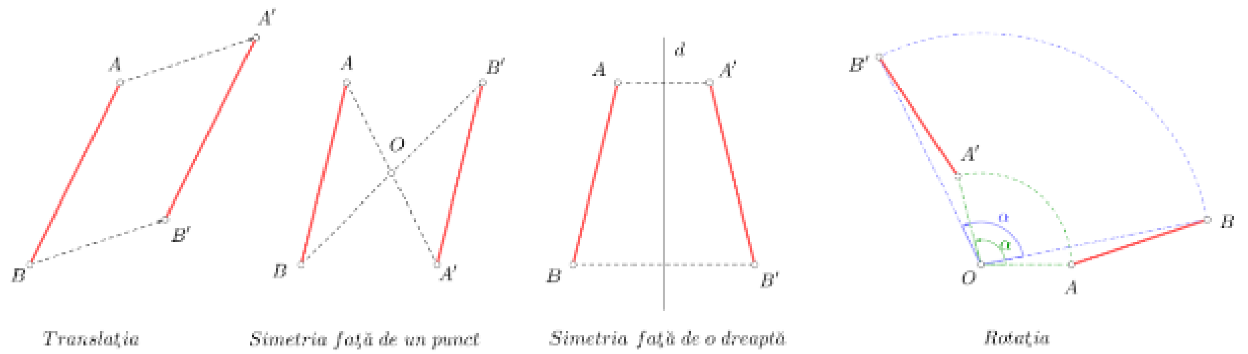
Students will be divided into groups of 3 and will answer the following questions:

1. How many types of friezes have been discovered?

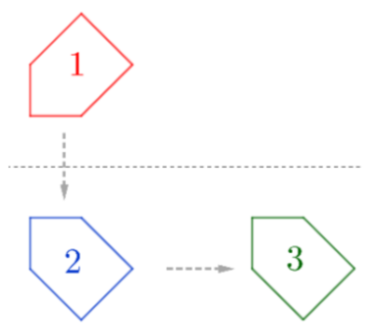
2. From a mathematical point of view, friezes are made by different .... (geometric transformations).

3. What are the isometries of the Euclidean plane?

Isometry is a geometric transformation that takes different points into different points and any segment into one congruent with it. The main isometries are translation, central and axial symmetries, and rotation around a point.



In addition, by composing several such isometries, an isometry is also obtained. For example: Glide reflexion is obtained by first applying an axial symmetry, followed by a translation parallel to the axis of symmetry.



Students are required to reproduce 2 or 3 of the seven types of friezes, depending on the isometries that leave them invariant. Their creations will be displayed in the class-room to be recognized by colleagues from the other teams.

Photo snapshots during class activities of Romanian students:

https://www.ciordas.ro/erasmus/erasmus20/frize/Friezes%2C%20common%20view1.pdf

https://stiribihor.info/2020/11/la-doar-23-de-ani-simon-nistor-duce-mai-departe-traditia-lazilor-de-zestre-de-la-budureasa/

Activity 2: Friezes using different geometric figures

Age: 12-16 years old

Equipment: computer, internet connection, coloured pencils, geometric tools, flipchart, YouTube documentation

Activity detail:

Different geometric figures are distributed to each group of students. If necessary, they will have coloured sheets and scissors to cut out other figures.

Students are asked to make, with the pieces provided, friezes from each of the seven existing types. The final product is photographed with a note specifying the changes made.

https://www.ciordas.ro/erasmus/erasmus20/frize/Friezes%2C%20common%20view1.pdf

https://www.ciordas.ro/erasmus/erasmus20/frize/Friezes%2C%20common%20view2.pdf

https://www.ciordas.ro/erasmus/erasmus20/frize/Friezes%2C%20common%20view3.pdf

Activity 3: Decorative friezes transposed into bookmarks with traditional patterns sewn on etamine

Age: 12-16 years old

Equipment: computer, internet connection, pencils, ideas taken from https://ro.pinterest.com, respectively from manual workbooks.

Activity detail:
- Under the coordination of the teacher of technological education, students learn the art of sewing, look for geometric patterns used in folk art and draw them on etamine. Bookmarks are made following the template.
- Different connections are made between the figures used on the bookmarks and the learned geometric notions, including the notion of frieze.

Activity 4: Friezes in folk art

Age: 12-16 years old

Equipment: objects and materials provided by folk craftsmen.

Activity detail:

Students have attended some workshops of folk craftsmen in the area and have admired the technique of shallow cutting in the wood mass, also called notching.

https://stiribihor.info/2020/11/la-doar-23-de-ani- simon-nistor-takes-away-the-tradition-of-the-dowry-boxes-from-the-budureasa

Friezes models on a traditional towel from Bihor

Extra activitiy in Neotrie VR

Make friezes in virtual reality making copies of a single figure and put them on the right positions after translatint, reflecting or rotating appropriately. or reflectint. One can also use the advanced tiling options of translation, mirror and rotation tools.
https://www2.ual.es/neotrie/project/geomview/#Friezes

Webography

https://fr-m-wikipedia-org.translate.goog/wiki/Groupe_de_frise?_x_tr_sl=fr&_x_tr_tl=ro&_x_tr_hl=ro&_x_tr_pto=nui,sc

https://ro.wikipedia.org/wiki/Friz%C4%83

https://books.google.ro/books?id=CHc6EAAAQBAJ&pg=PA24&lpg=PA24&dq=Friza,+matematica&source=bl&ots=rEkQSg6aek&sig=ACfU3U1dyeyoJq7KEj6uqOoSU-BozmZpt3Q&hl=ro&sa=X&ved=2ahUKEwj24q3brv_zAhWZ_7sIHSvuBDIQ6AF6BAg-MEAM#v=onepage&q=Friza%2C%20matematica&f=false

https://ro.pinterest.com/pin/425097652318253319/

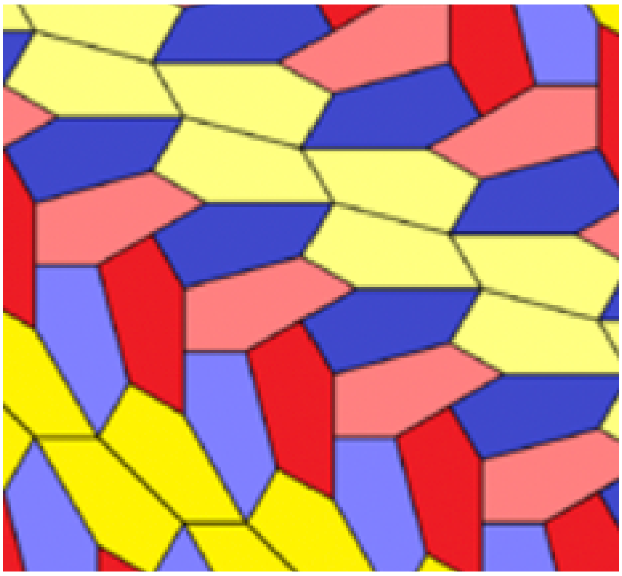
https://motivector.ro/

Artistic consultants, teachers Liana and Pavel Bența

Theoretical support provided by Conf. univ. dr. Fechete Ioan, University from Oradea, Mathematical and Sciences Faculty.

Tessellations

Mathematics, Nature, Science, Art = Beauty



A tessellation or tiling is the covering of a surface, often a plane, using one or more geometric shapes, called tiles, with no overlaps and no gaps. In mathematics, tessellation can be generalized to higher dimensions and a variety of geometries. A periodic tiling has a repeating pattern.

Tessellation in two dimensions, also called planar tiling, is a topic in geometry that studies how shapes, known as tiles, can be arranged to fill a plane without any gaps, according to a given set of rules. These rules can be varied. Common ones are that there must be no gaps between tiles, and that no corner of one tile can lie along the edge of another. There are only three shapes that can form such regular tessellations: the equilateral triangle, square and the regular hexagon.

Anyone of these three shapes can be duplicated infinitely to fill a plane with no gaps.

Educational goal

Students and teachers can use these constructions: to study a fractal, to learn how geometrical properties remain while the new shape progressively become smaller, to express the mathematical characteristics with the help of geogebra application, to use mathematics as a building block in constructions

Target Age

Students and teachers

Skill required

Geometry (Area, perimeter, properties, theorems)
Geogebra skills (create shapes, create a tool, using commands, platform of geogebra)

Mathematical and artistic goal

Students adopt new skills, drawing, imagining
Students have the challenge to make impressive images and constructions

Learning activities

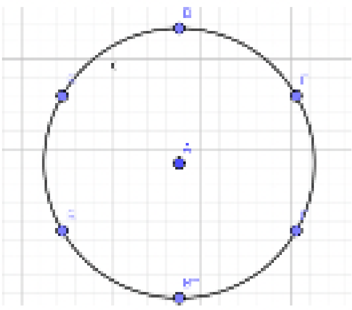
Activity 1: Creation of a 6-star Islamic art

Age: 15-18 years old

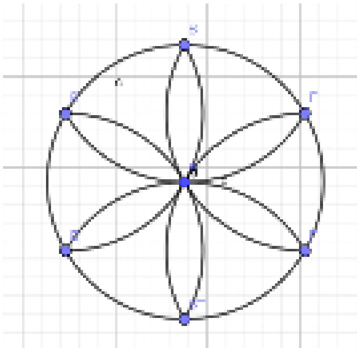
Equipment: pencil, calculator, online notebook for using GeoGebra https://www.geogebra.org/geometry

Activity detail:

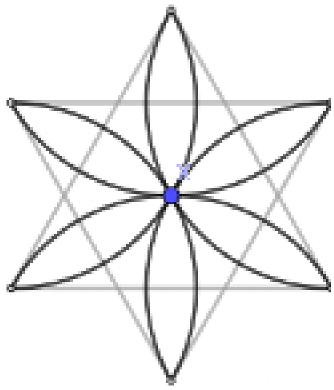
Steps:



Create a slider n (integer) Also create a circle (A,n) using the tool: circle with centre through point Point B on the circle. Using the tool: Angle with given size, select B leg point , A, the vertex, 60º angle We divide the circle into 6 equal arcs



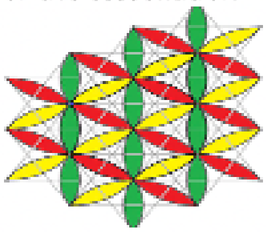
Tool : Circular arc, we select the point B as the centre and two consecutive points on the circle We repeat the procedure

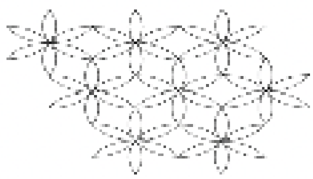


Create the two above equilateral triangles

Create a new tool
Output objects: all the arcs and the points.
Input objects: the point A, the slider n
Apply the tool1:
Select the new tool, select a point on the circle, type the letter n
Create a new tool
Output objects: all the arcs, the triangles, all the point vertexes. Input objects: the point A, the slider n
Apply the tool2:
Select the new tool2, select a point on the circle, type the letter n

Color the tessellation





Activity 2: Creation of a 8-star Islamic art

8-star

Slider n

Create a circle (A,n) , Tool circle with center and radius

Point B on the circle

Tool: Angle with given size: Select B as leg point, center A, angle 45° (we divide the circle into 8 equal arcs)

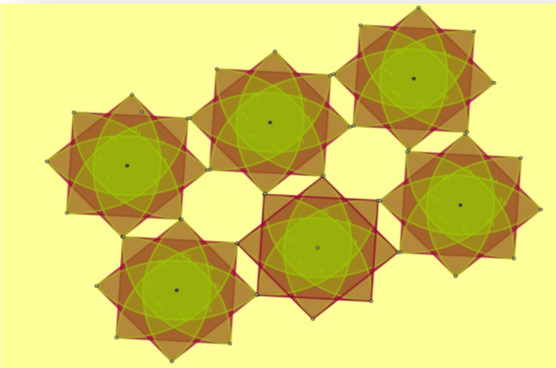
circle with center through points

Select point B as the center and two consecutive points on the circle We repeat the procedure

Create a new tool

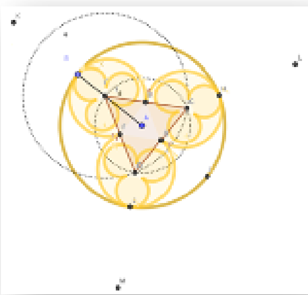
Output objects: all the arcs, the squares, all the point vertexes Input objects: the point A, the slider n

Apply the tool:
Select the new tool, select a point on the circle, type the letter n

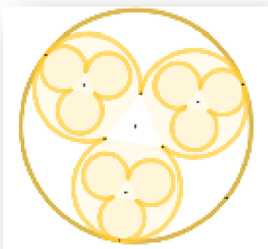


Activity 3: Creation of a gothic art

R= slider
Big circle(A,R), B point on the circle
Small circle (A,r),AΓ= r=√3/3, Γ point Γ on this circle
Construct an equilateral triangle ΓΔΕ , ΓΔ=R√3



Tool: Circumcircular arc
Select the points Δ,Η,Ζ
Η, Ζ are midpoints
Do the same for
Γ,Η,Θ and for Ε,Θ,Ζ
Create the circle (A,AB)



Create a new tool
Tool1
Output objects: all the arcs,
Input objects: the point A, the slider R

Apply the tool:
Select the new tool1, select the point Γ , type the number ΒΓ. Apply the tool on the points C and D

AH=HL, AB=BK, AJ=JM

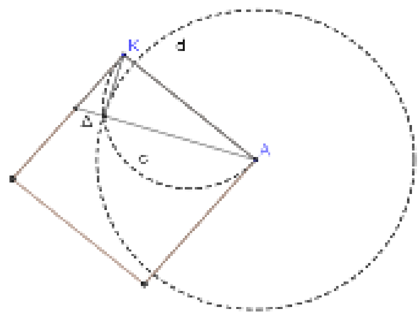
Create a new tool
Tool2
Output objects: all the arcs, the points L, K,M, the circle (A,AB)
Input objects: the point A, the slider R

Apply the tool:
On the points K,L,M
giving the number R

Activity 4: Creation of a Pythagora’s Tessellation

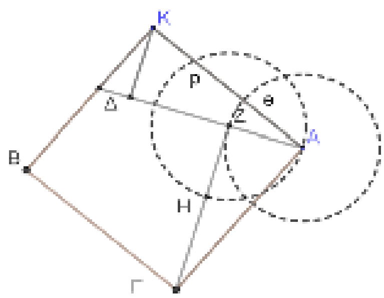
Sliders n, i   integers
In the field of Algebra type
ypo=(i² + n²)^0.5

Create circle (K, ypo)
KA radius of the circle. Create a square with side KA. Create a circle (K,n), create a semicircle with diameter KA, Δ the intersection point. The triangle KΔA is right



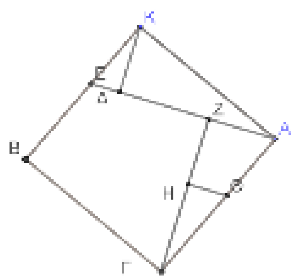
Create circle (A,i), the intersection point with segment AΔ is Z
Create circle (Z,i), the intersection point with segment ZΓ is H

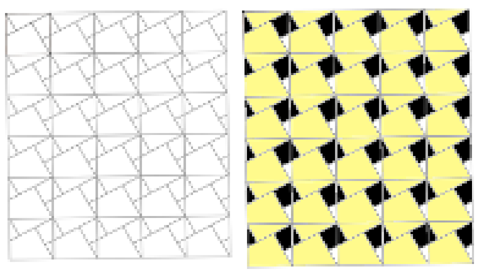
HΘ//AΔ
Tool polygon: Construct three triangles and two quadrilaterals



<u>Create a new tool</u>

Output objects: all the triangles, the quadrilaterals, all the point (vertexes, etc.)
Input objects: the point K, the sliders n, i





Create a new tool
Tool1
Output objects: all the arcs,
Input objects: the point A, the slider R

Apply the tool:
Select the new tool1, select the point Γ , type the number BΓ. Apply the tool on the points C and D

AH=HL, AB=BK, AJ=JM

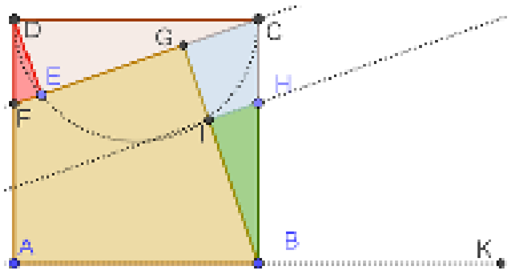
Create a new tool
Tool2
Output objects: all the arcs, the points L, K,M, the circle (A,AB)
Input objects: the point A, the slider R

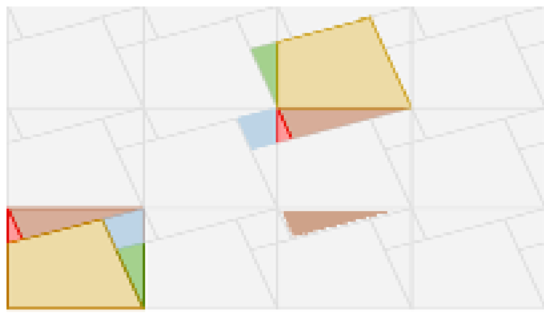
Apply the tool:
On the points K,L,M
giving the number R

Activity 4: Creation of a Pythagora's Tessellation

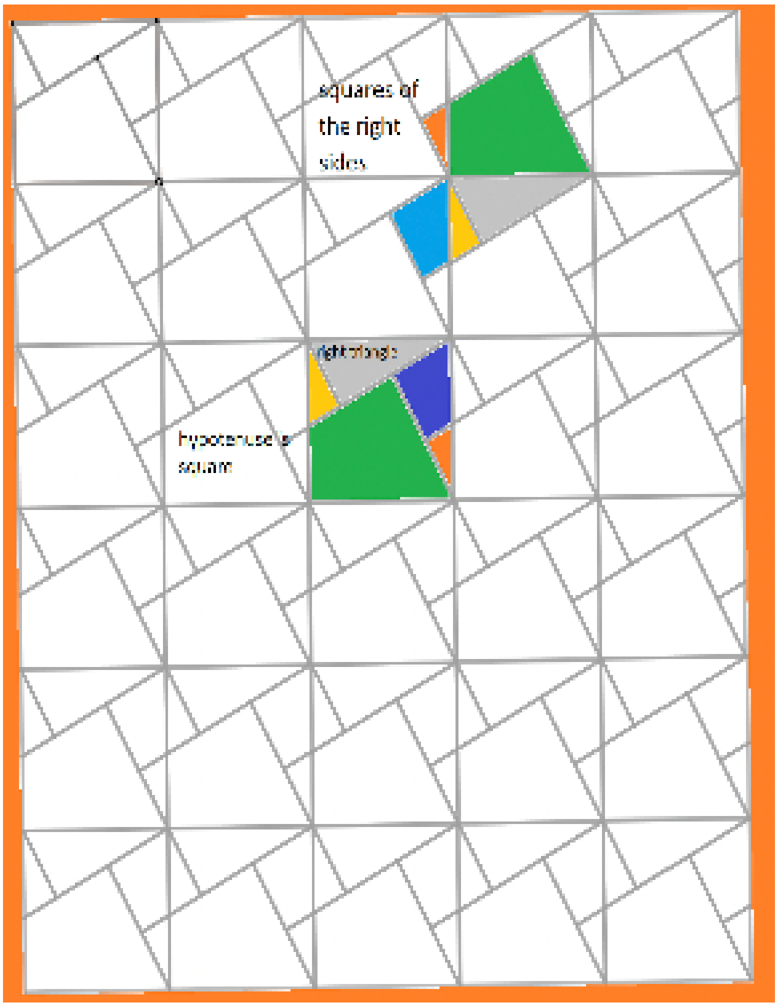
Another easier construction

We create:
a square ABCD, a right triangle DEC, segments CG=DE, CH=DF, HI//CG. We form the polygons : DEF, CHIG, BHI, BGFA. This shape will be our basic tile for the tiling. We also create the point K such us A,B, K to the same line and AB=BK (for better use of the tool will be created)WE create a new tool, output objects : the points K,D,C and the 5 interior polygons. We apply the new tool and we create the tessellation





Prove of Pythagorean theorem:



The colors area-pieces consists two formations. One is the the square whose side is the hypotenuse (the side opposite the right angle). The other sum of the areas of the squares on the other two sides

Extra activitiy in Neotrie VR

Play with the Pythagorean tessellation in virtual reality.
https://www2.ual.es/neotrie/project/geomview/#Pythagorean tiling

Our work

Tessellations hexagon: https://www.geogebra.org/m/cyvfaaj9
6 star: https://www.geogebra.org/m/mvffv2ey
8 star: https://www.geogebra.org/m/hpdmvj96
Gothic: https://www.geogebra.org/m/pkjk6nqp
Gothic filled: https://www.geogebra.org/m/ayfsrgs2
Pythagora's till: https://www.geogebra.org/m/jjzs8xjs

Webography

https://en.wikipedia.org/wiki/Tessellation#:~:text=A%20tessellation%20or%20tiling%20is,tiling%20has%20a%20repeating%20pattern.
https://www.mathsisfun.com/geometry/tessellation.html
https://www.youtube.com/watch?v=YompsDIEdtc

Spheres

Mathematics, Nature, Science, Art = Beauty

A sphere is the set of all points in 3D space lying in the same distance r (the radius) from a given point (the center). It is a surface of a ball.

A ball is a 3D shape inside of a sphere including all of sphere's points.
In common language, people identify the ball as the sphere. So, we can say that a sphere has its surface and volume.

Surface area: $A = 4r^2\pi$

Volume: $V = \frac{4}{3}r^3\pi$

Word «sphere» comes from a Greek word **σφαῖρα** — sphaira, meaning «globe, ball».

Educational goal
Introduction to sphere theory and the calculus of sphere's surface area and volume

Target Age
Students aged from 15 to 19

high school

Skill required
Basic knowledge of geometry and calculus

Ability to draw, use technology and take pictures

Mathematical and artistic goal
look at sphere from an artistic «angle» (use advanced artistic ways of drawing a sphere)

Learning activities

Activity 1:

Students measure spheres of different sizes and calculate their surface areas and volumes and they construct proportional spheres using GeoGebra (or NeoTrie)

Students artistically color those spheres (real and virtual)

Equipment: tailor's meter, thread, ruler, spheres of different sizes, PC, GeoGebra

Activity detail:

Students will get various spheres and necessary tools with which they will measure the circumference of those spheres and calculate the radiuses. After that, they will calculate surface areas and the volumes of those spheres and artistically paint them and arrange them into some theme (for example a Solar system). Besides, they will use 3D GeoGebra and construct proportional spheres colored by different colors.

https://bit.ly/solar_system_geogebra (student's poster)

Activity 2:

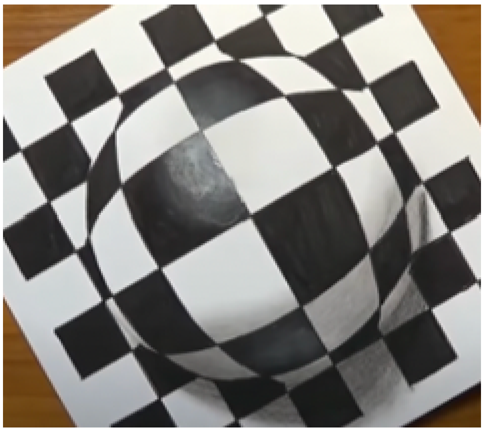
Students draw a 3D Sphere on a 2D paper (spherical chessboard)

Equipment: white or colored paper (with no lines), pencil, ruler, felt-tip pen, marker, white pencil or chalk, and a handkerchief.

Activity detail:

Students draw straight and curved lines which intersect and thereby make a net of cells that they color in a way that they get a chessboard.

https://www.youtube.com/watch?v=NZfj81PEn9o
https://bit.ly/spherical_chessboard (student's poster)



Activity 3:

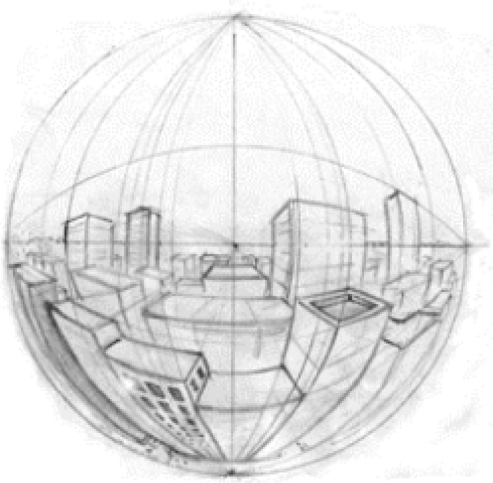
Students learn about the techniques of 3D drawing with special emphasis on technique called Five-point perspective

Equipment: white paper, pencil, divider, ruler

Activity detail:

Students learn the technique of drawing named Five-point perspective using links below and apply that technique; they arrange the drawings according to their artistic will and abilities or using inspiration they can find on the Internet

https://improveyourdrawings.com/2019/10/learn-the-types-of-perspective-in-drawing/
http://becktastic.weebly.com/fish-eye-5-point-perspective.html
https://www.deviantart.com/awlaux/art/Five-Pt-Perspective-Tutorial-96369849
https://bit.ly/5_point_perspective (student's poster)



Activity 4:

Students take artistic photos through a glass sphere

Equipment: glass sphere, camera (or smartphone)

Activity detail:

Students take artistic photos through a glass sphere and notice that the image through a glass sphere is twice reversed (up-down and right-left); they learn why does that happen.

https://www.youtube.com/watch?v=ohnaQRid2sc

https://bit.ly/Glass_sphere_Optics (student's presentation)

Extra activitiy in Neotrie VR

Construct a scaled copy of the Solar system in virtual reality.
https://www2.ual.es/neotrie/project/geomview/#Solar system

Webography

https://www.youtube.com/watch?v=NZfj81PEn9o
https://improveyourdrawings.com/2019/10/learn-the-types-of-perspective-in-drawing/
http://becktastic.weebly.com/fish-eye-5-point-perspective.html
https://www.deviantart.com/awlaux/art/Five-Pt-Perspective-Tutorial-96369849
https://www.youtube.com/watch?v=ohnaQRid2sc

Results

Solar system and GeoGebra:
https://bit.ly/solar_system_geogebra

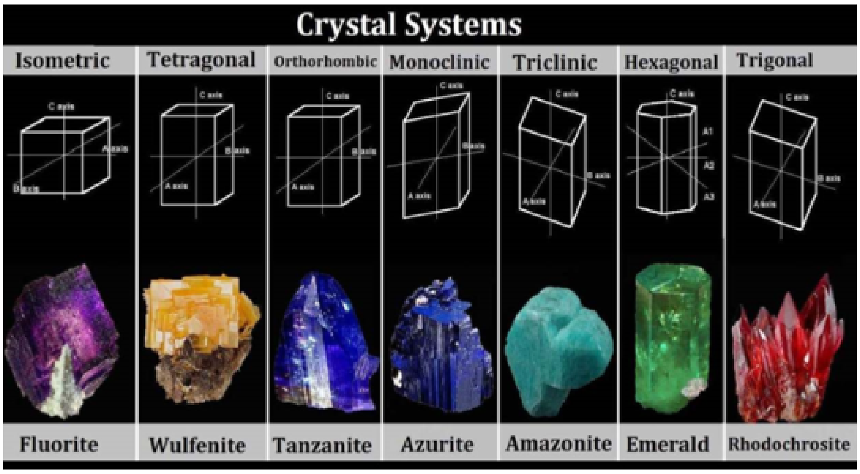
Spherical chessboard:
https://bit.ly/spherical_chessboard

5 points perspective:
https://bit.ly/5_point_perspective

Glass sphere:
https://bit.ly/Glass_sphere_Optics (student's presentation)
https://bit.ly/Glass_sphere_optics_poster (student's presentation as poster)
https://bit.ly/GV_Sphere_glass (poster - photos)

Crystals

Mathematics, Nature, Science, Art = Beauty



What is a snowflake?

When people say snowflake, they often mean snow crystal. The snow crystal is a single crystal of ice, within which the water molecules are all lined up in a precise hexagonal array. Snow crystals display that characteristic six-fold symmetry we are all familiar with.

What is a crystal?

A crystal is a physical body in a solid state that is built of properly three-dimensionally periodically arranged atoms, ions, or molecules, which means that it has a crystalline structure. If the growth of crystals took place without external impurities, their external forms are geometrically regular (polyhedra), which is a reflection of their regular structure. Crystallography is the science of crystals. It was developed because of mineralogy and its theory relies on physics and mathematics.

Educational goal

introduction to theory of Crystal systems

Target Age

students aged from 14 to 19 high school

Skill required

developed motor skills
spatial vision

Mathematical and artistic goal

work with various accessories and materials

Learning activities

Activity 1:

Students learn about seven crystal systems and make 3D models of them

Equipment: wooden sticks, styrofoam balls (or some other material), glue, paint, paper models, scissors

Activity detail:

The Chemistry teacher gives students a short lesson about seven crystal systems. Students afterward explore the theme and make models of those seven crystal systems (with wooden sticks, styrofoam, and paint, or with paper models, scissors, and glue).

https://www.periodni.com/download/models_of_crystal_systems.pdf
https://www.indigoinstruments.com/molecular_models/unit/kits/7-crystal-systems-models-69107.html
https://bit.ly/GV_Crystals (Teacher's poster)

Activity 2:

Students make models of seven crystal systems using GeoGebra or NeoTrie.

Equipment: PC, GeoGebra, Meta quest 2, NeoTrie

Activity detail:

Students are divided into seven groups and every group picks one crystal system and makes its model using GeoGebra or Neotrie.

https://www.thingiverse.com/thing:749332
https://www.geogebra.org/m/rxvmc3uj
https://www2.ual.es/neotrie/project/geomview/#Crystals

Activity 3:

Students make real crystals of sugar, salt, citric acid etc.

Equipment: Sugar, salt, citric acid (or something else), water, pot, jars, wooden sticks, tongs, pencil, paper,...

Activity detail:

Students will on their own make various crystals (of sugar, salt, citric acid, ...) and in those, they will notice various geometrical shapes. Students will take photos of their work and posters of those photos.

https://youtu.be/jzRyL3oZnro
https://www.pinterest.com/pin/414542340709145014/
https://bit.ly/snowflakes_crystals_photos (student's photos)

Activity 4:

Students make jewelry out of artificial crystals

Equipment: artificial crystals, plastic flax or thread or rubber thread or metal wire, pliers

Activity detail:

Teachers will secure needed materials and give it to students who will make jewelry out of artificial crystals (bracelets, earrings, ...)

https://www.youtube.com/watch?v=xaPswV4EHjY

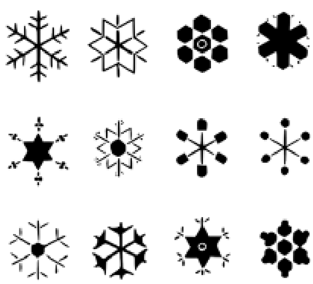
Activity 5:

Students explore about snowflakes, make paper models and posters, take photos

Equipment: Internet, PC, paper, scissors, protractor, camera

Activity detail:

Students will explore the history of snowflakes' researches, and make presentations and posters. Through their research they will learn about the various shapes that snowflake can take and they will make proper paper models of snowflakes. Also, they will take pictures of real snowflakes and make posters of their photos.



https://www.firstpalette.com/pdf/snowflake1.pdf
https://www.youtube.com/watch?v=1wbTyGJfbJs&t=264s
https://bit.ly/GV_Snowflake (student's poster)
https://bit.ly/Geometry_of_snowflake (student's presentation)
https://bit.ly/snowflakes_models (student's poster)

Extra activitiies in Neotrie VR:

Build a crystal from its fundamental cell and fly through geant crystal structrues, discover hidden symmetries with the scanner tool.
https://www2.ual.es/neotrie/project/geomview/#Crystals

NeoTrie VR

Webography

https://www.periodni.com/download/models_of_crystal_systems.pdf
https://www.indigoinstruments.com/molecular_models/unit/kits/7-crystal-systems-models-69107.html

https://www.thingiverse.com/thing:749332
https://www.geogebra.org/m/rxvmc3uj
https://www.youtube.com/watch?v=xaPswV4EHjY
https://www.firstpalette.com/pdf/snowflake1.pdf
https://www.youtube.com/watch?v=1wbTyGJfbJs&t=264s

Results

Crystals:
https://bit.ly/GV_Crystals
https://bit.ly/crystal_models

Snowflakes:
https://bit.ly/Geometry_of_snowflake
https://bit.ly/GV_Snowflake
https://bit.ly/snowflakes_models

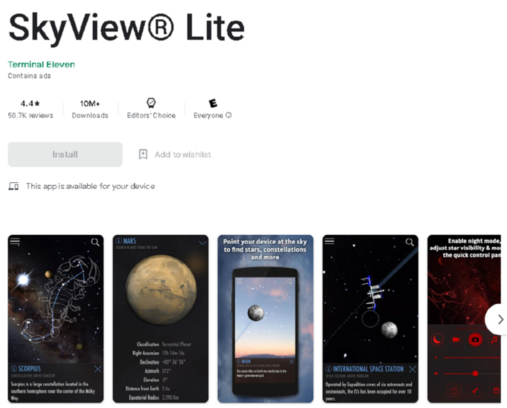
Crystals&snowflakes:
https://bit.ly/snowflakes_crystals_photos

Constellations

Mathematics, Nature, Science, Art = Beauty

Figure 1: Capture of SkyView

How can space be explored in the classroom by connecting technology, mathematics and art?



The following three activities show how space topics can be explored in the classroom by connecting technology, mathematics and art.

Educational goal	Target Age	Skill required	Mathematical and artistic goal
Learn about real life application of mathematics in astronomy	Students of 13-15 years old. Lower secondary school.	Use of application on mobile phone	Learn about Cartesian coordinates; Use space as an artistic inspiration

Learning activities

Activity 1:

We all know that we need a telescope to explore space, but there are many apps that could provide a glimpse of the sky to students. For that purpose we present the SkyView® Free app for android to explore sky and constellations in the sky. It works without wifi also. In Figure 1 the app logo can be seen.

Students could download this free application to their phones and observe the sky during the day or in the evening to stargaze stars and to comment on what they have seen and noticed.

The app uses a phone camera and identifies celestial objects in the sky. Students could explore constellations by scanning across the sky, and also locate planets in the solar system, satellites and galaxies by pointing their devices at the sky. The app has night and day modes, so the observation can be made during the day and during the evening. After students spot the objects, with the augmented reality feature of the app, they should talk about what they have seen.

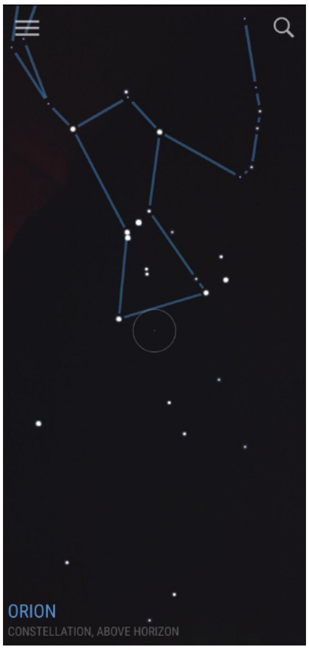


Figure 2: Students captured the constellation of Orion

Figure 3: Students captured the Big Dipper

For example, it is easy to find the Big Dipper, which is the part of Ursa Major, and that can help students to find the Little Dipper which is part of Ursa Minor. Also, students can find Orion or so called Hunter, where three bright stars make this constellation very easy to find. It can be very interesting to students to observe constellations such as Taurus or Gemini and connect to their date of birth and horoscope.

Students can follow the sky tracks for objects and explore locations in the sky, but also they can experiment with the different dates and times. In order to find inspiration for artwork, students should be encouraged to capture very nice images. In Figure 2 we can see a capture during sky observation. It is a capture of the Orion constellation, while in Figure 3 it is the Big Dipper captured.

Activity 2:

In order to connect mathematics and sky observation, we have chosen an activity where students learn about plane coordinates. There are several objectives for students such as: to know how to draw and label the Cartesian plane and its parts, to plot points on the point, to read coordinates for a point for a graph, and to use GeoGebra to plot points.

In the begining of this activity students learn about the Cartesian plane.

The Cartesian plane is named after the French mathematician and philosopher Rene Descartes. It is defined as two perpendicular numbered lines. It is common to call the horizontal line as x-axis and vertical line as y-axis. Using x and y axes every point can be described by use of ordered pairs of numbers. For example, the point A (-2, 1) is 2 units to the left and 1 unit up on the coordinate system. In order to plot a point on a graph students first need to identify x and y axes, locate values on the x and y axes, and plot the point where x and y meet.

Students' task is to plot the Orion constellation. One part of stars in this constellation are: Betelgeuse, Alnitak, Salph, Alnilam, Mintaka, Bellatrix and Rigel. Their coordinates are (-5,9), (-2,-1), (-3,-9), (0,0), (2,1), (5,8), (7,-7), respectively. The solution is shown in Figure 4. This solution is done by hand, and in Figure 5 we can see the task's solution in Geogebra.

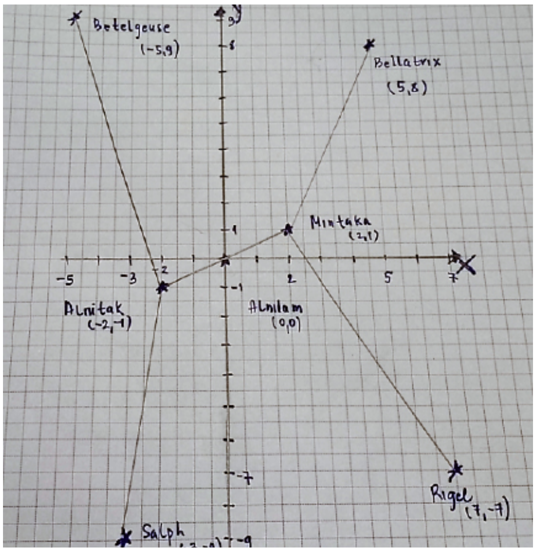


Figure 4: Plotted Coordinates of some stars in the Orion constellation

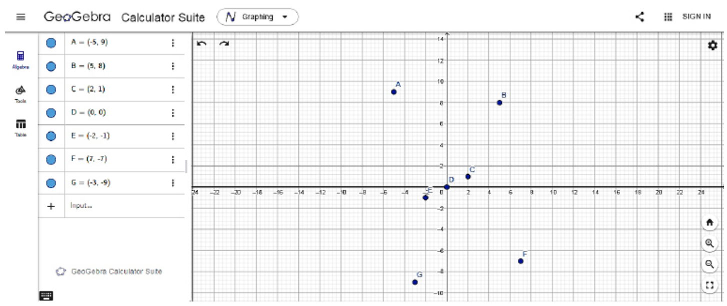


Figure 5: Plotted Coordinates of some stars in the Orion constellation in Geogebra

The second task was to find the coordinates of points that represent the Big Dipper. In Figure 6 we can see the task given to students.

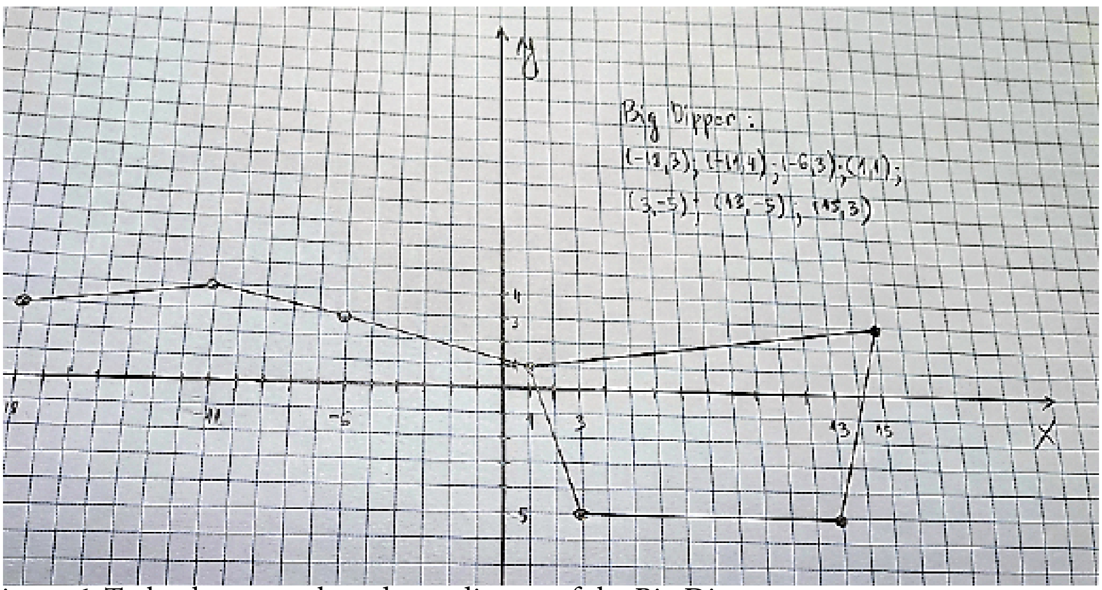


Figure 6: Task where are plotted coordinates of the Big Dipper

Activity 3:

This activitiy shows how space can be used as an inspiration in artwork. This finally summarizes topics of previous activities but through artistic design.

Students get the opportunity to explore various techniques in order to express themselves about constellations.

Art teachers explain to them how to achieve certain effects on their pictures, such as shadows or lights. Some of the ideas and students results are here shown.

Extra activitiy in Neotrie VR:

Understand that the stars of a constellation that we see projected on the celestial sphere are actually far away from each other in depth. This is shown with an example in the Ursa Major in virtual reality:
https://www2.ual.es/neotrie/project/geomview/#Constellations

Webography

https://spacemath.gsfc.nasa.gov/stars/10Page67.pdf
https://spacemath.gsfc.nasa.gov/IRAD/IRAD-1.pdf
https://spacemath.gsfc.nasa.gov/stars.html
http://www.tsgc.utexas.edu/space_vision/constellations.pdf

3D Printing

Mathematics, Nature, Science, Art = Beauty

How can we provide students with skills required for the future? Working on 3D printing projects, including 3D modeling, improves creativity, technology literacy, problem-solving, self-directed learning, critical thinking, and perseverance.

The practical application of 3D printing and its technological advances impact different fields in everyday life.

By introducing 3D printers to students we are giving the foundations for improving this technology in the future, which might improve our quality of life and make many things more accessible.

Quick replication of objects has the possibility to improve lives in the future! Some predictions for the 3D printing future can be seen in the car above (see also Figure 1).

Educational goal	Target Age	Skill required	Mathematical and artistic goal
Modeling ideas into real life subjects	Students of 14-16 years old	Use of 3D printer	Exploring mathematical symbols; design of jewelry

Learning activities

Activity 1: Introduction to the 3D printing

3D printing process starts from an idea and continues to modeling, which results in producing a model. This process might be very helpful in conceptualization and visualization many STEM concepts.

The usage of 3D printers in the classroom combines science, technology, engineering, math, but also has aesthetic and artistic aspect. That makes 3D printer excellent tool for supporting STEM+art concept which combines to empower students in the developing competences such as inquiry or critical thinking.

Activities during 3D printing are close to the reality, usually based on solving real-world problems.

Figure 1: Future of 3D printing

3D printing activities and projects could help to the conceptual understanding of different things and to self-guided inquiry and this lessons could be applied with the students age 13-16.

Preparation for 3D printing can be done in use with various programs such as GeoGebra (www.geogebra.org), or Sketchpad (https://sketch.io/sketchpad/). There are several stages of 3D printing prior to the final product.

Steps for preparation of 3D printing file by using GeoGebra can be seen next. In Figure 2 we can see a tetraedron made by origami techniques.

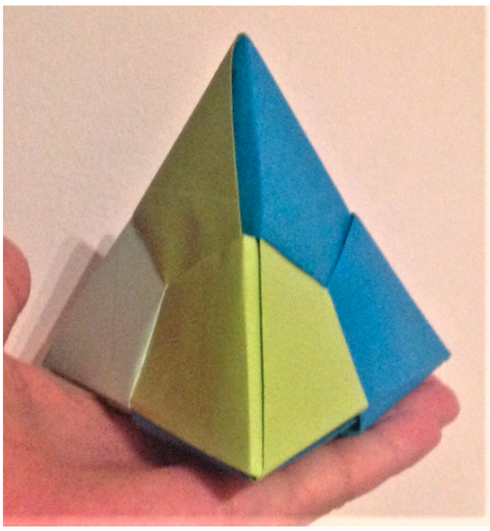


Figure 2: Tetrahedron made of paper by origami tech

This tetrahedron can be modeled in GeoGebra as it is shown in Figure 3. Figure 4 shows how can GeoGebra file be transferred to STL file which is suitable to 3D printing. (see also extra activity in Neotrie VR below).

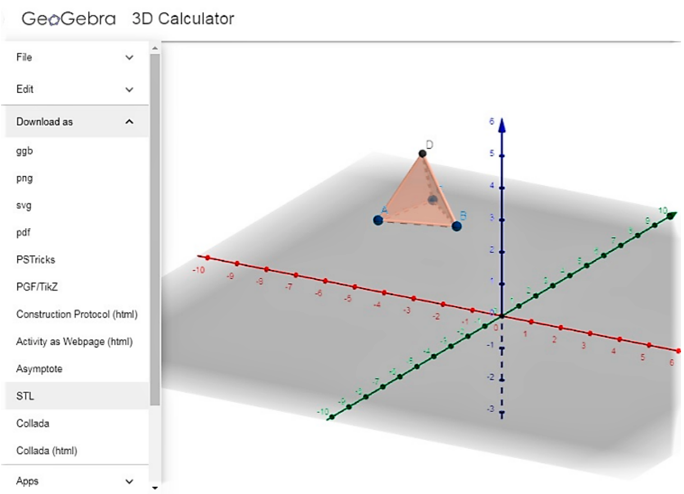


Figure 3: Tetrahedron modeled by GeoGebra

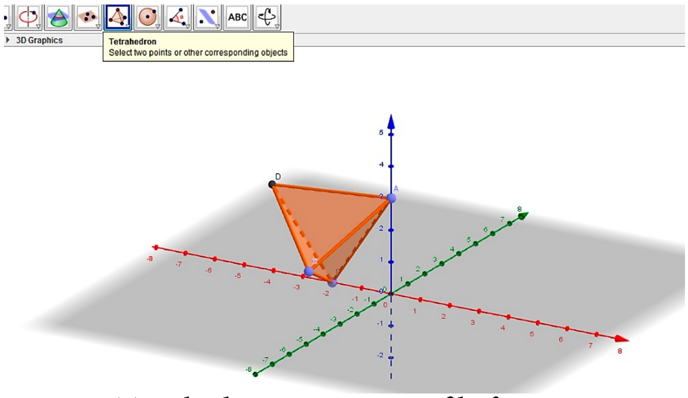


Figure 4: Tetrahedron preparation file for 3D printing

Activity 2: 3D printing of prisms

In this activitiy students had to apply their knowledge related to the proportions and make a model of prism that would resemble to the Lego brick. In this process students explore more information about Lego bricks, from the interesting playful facts to the mathematical facts and proportions. Collaborating together, students get 3D printed models of Lego bricks as it is showed in Figure 8.

The Lego Group began its manufacture of the interlocking toy bricks in 1949. Many movies, games, competitions and Legoland amusement parks have been developed under the brand. As of July 2015, 600 billion Lego parts had been produced.

First step would be explore Lego education resources which help students to see how incredible things could be made with help of Lego. More about Lego educational sources could be found at https://education.lego.com/en-us/.

Information about this great toy suitable for all ages brings joy to children and to those who «never grow up». Students can make a collaborative «Lego facts exhibition» in the classroom about Lego and facts related to them.
Lego elements hide their own geometry, some of it is very simple, such as three plates stacked on top of each other are as high as a standard Lego brick. Also, it is interesting that stacking five plates on top of each other makes them as high as the width of a 1x2 brick.

Different kinds of Lego bricks have different kinds of geometry. For example, the headlight bricks become even more interesting if you discover their own geometry.

Next step would be researching about hidden math in Lego brick form, since it has prismatic form. One of the main idea is to explore proportions of the bricks, since bricks are mathematical notion. Lego bricks has its own geometry and there is lot of proportions.

More about Lego brick geometry can be found https://holgermatthes.de/bricks/en/geometry.php

Proportion of the Lego bricks is known as 5:6, which can be seen in the Figure 5.

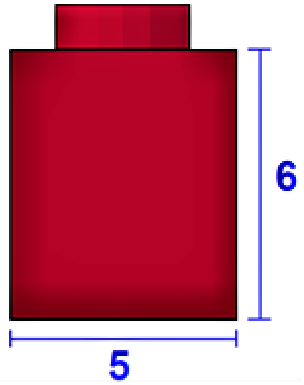


Figure 5: Lego bricks ratio

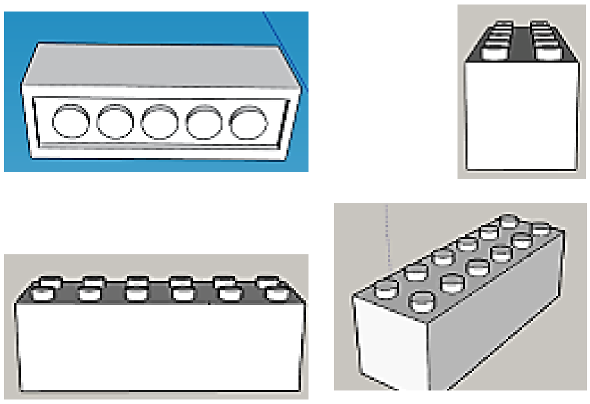


Figure 6: Lego brick modeled with use of Sketchpad

After exploring Lego bricks properties students model it in order to prepare files for 3D printing. For example, they can use Sketchpad, as it is show in Figure 6 below. Sketchpad https://sketch.io/sketchpad/ is a free online drawing application, where digital art work can be shared or export in different formats.

Sketchpad files can be prepared for 3D print with software Cura or EasyPrint 3D. Process of preparing for 3D print is shown in Figure 7 below.

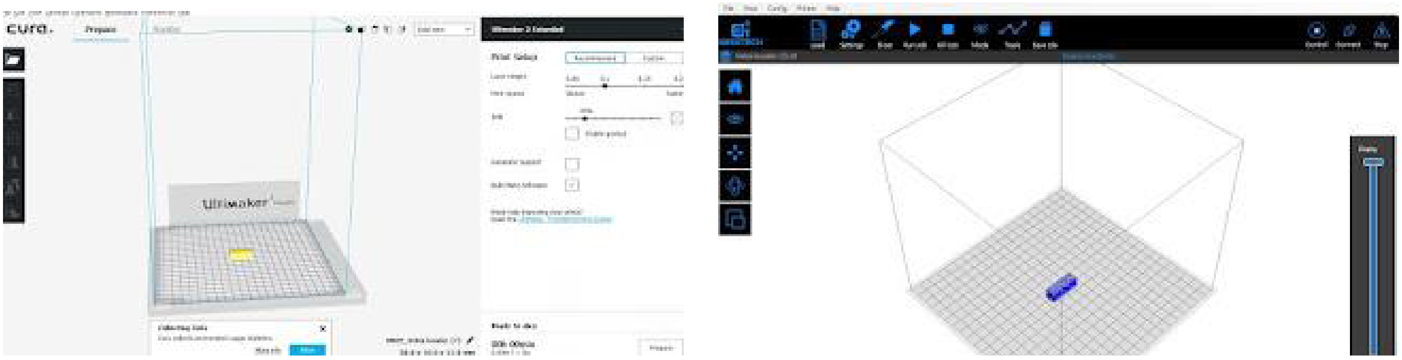


Figure 7: Process of preparing for 3D print in Cura software

Depending on the 3D printer quality of printer the outcome quality could vary. One of the possible result could be seen in Figure 8.

Who knows, one day some of our student will become an engineer or designer in a Lego factory.

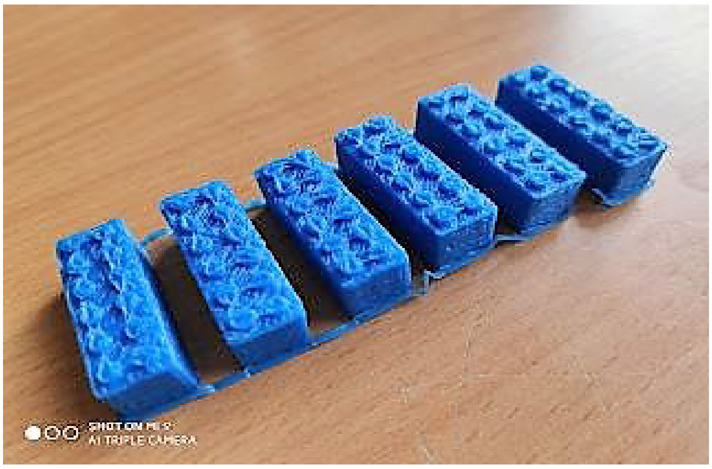


Figure 8: 3D printed Lego cubes (work of a student)

Activity 3: 3D printing of jewelry

In this activity students can explore prepared files for 3D printing and use to explore mathematical concepts. Beside printing mathematical objects and geometrical solids, students can 3d print something nice and useful.

Our idea was to print pendant as a piece of jewelry and give as a present someone. For example, students can chose the «tree of life».

Tree of life is known as a popular symbol that represents different things in various cultures. It is used all over the world, although it is not specific for a particular culture and have different meanings such as ancestry, family, fertility, growth, strength, individuality, immortality, rebirth or peace.

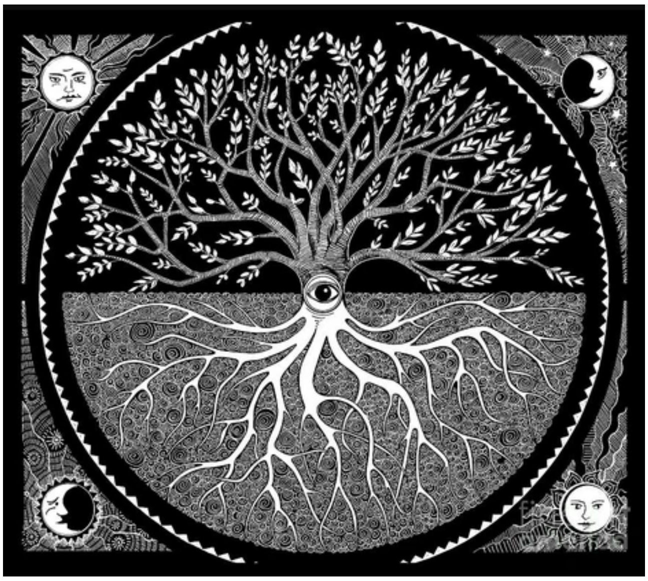


Figure 9: Tree of life

Usually, it is represented as a large tree with branches and roots that spread, reaching up in the sky and below the ground. Many times the «tree of life» is contained within a circle. Due to that, many people, since ancient times from Egyptians, Assyrians or Persians, like to wear that symbol as a piece of jewelry. Beautiful attributes that symbolize: knowledge, wisdom or focus bring timeless beauty. One of the representation can be seen in Figure 9 (https://bhira.ch/blogs/blog/the-meanings-of-the-tree-of-life).

Beside artistic beauty, tree has a mathematical aspect. By mathematical definition, a tree is a set of nodes that are connected by edges in such a way that do not exist more than one path between any two nodes.

Also, tree is used to illustrate the evolution of life on Earth, where each branch represent the evolution of a new species. Termination of the branch represents the extinction of that species. The shape of the tree seems to be repeated on smaller and smaller scales, which is, in mathematical meaning a fractal.

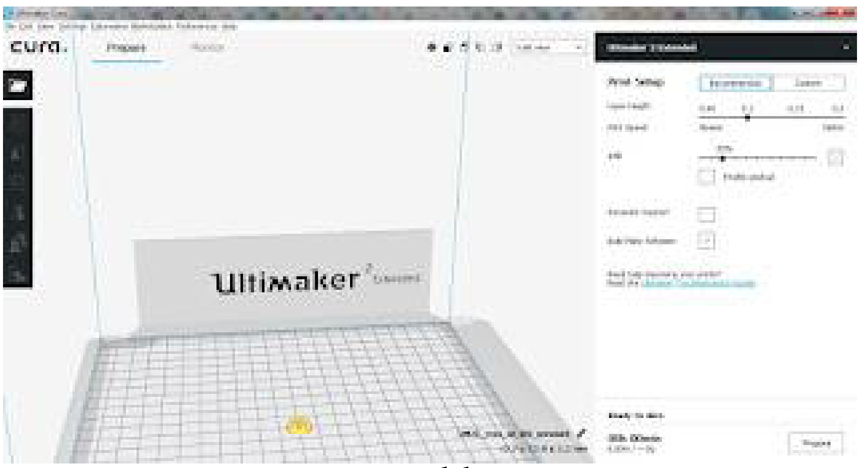


Figure 10: Modeling process

One example of the «tree of life» design can be found at www.thingiverse.com/.

The preparation looks like in Figure 10. Printing process can be seen at Figure 11 and final result is on Figure 12.

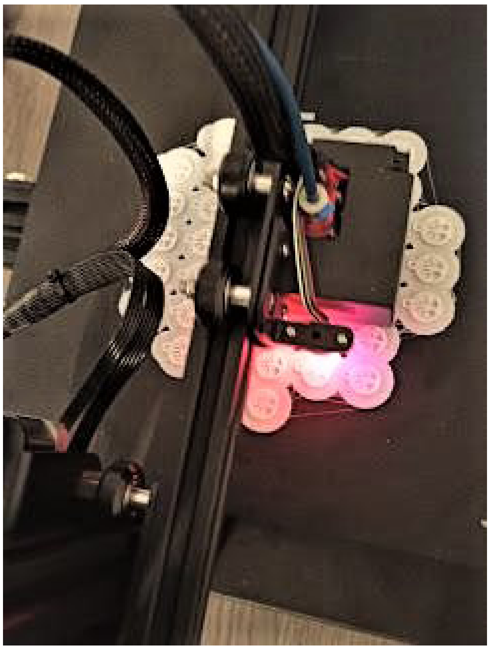


Figure 11: 3D printing process

Figure 12: Final result of 3D printing

Extra activitiy in Neotrie VR:

Use the STL importing, editing and exporting options in virtual reality.
https://www2.ual.es/neotrie/project/geomview/#3d printing

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