



Co-funded by the  
Erasmus+ Programme  
of the European Union

# Brochure



***Explore the mathematical  
culture through arts and  
sciences***

**Project number: 2020-1-FR01-KA227-SCH-095534**

**2021-2023**

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# Project presentation

The Erasmus + project KA225-SCH-095534, Geometrician's views (GV) is an international cooperation in the field of education. The aim of the initiative is to enable pupils from kindergarten up to high school level to explore mathematical culture through art and science.

Partners from 7 different countries are invited to exchange innovative approaches towards teaching maths, students are invited to create collaborative deliverables at the intersection between mathematics and arts.

The initiative includes an international staff training open to the members of the participating organisations, as well as multiplier events to disseminate the project outcomes.

The GV consortium expects to produce a number of tangible outputs including supports for traveling exhibitions on maths and arts, guides for the teachers who wishing to implement hands-on activities on this topic in the classroom.

The strategic goals are:

- To disseminate project activities, findings and outcomes to stakeholders (NGOs, educational institutions, instructional environments, etc.).
- To identify and engage stakeholders throughout the project in order to widen the project stakeholder base.
- To facilitate and share understanding of educational approaches suitable for formal activities on maths and arts.
- To maximise the impact of the outcomes of the project (the educational resources, publications, etc.).
- To raise awareness of innovation in education, especially in mathematics, among the general public.

In addition, this strategy comprises a more practical objectives:

- To motivate partners to communicate about the project on social media and online platforms.
- To prepare communication supports (flyers, posters, banners, etc.) and other materials for dissemination.

Website:

<https://geometricians-views.lesmathscene.fr/>



# ● The partners ●



ITALY: Istituto Comprensivo Perugia 14

SPAIN: Virtual Dor

SERBIA: Petro Kuzmjak school

ROMANIA: Middle school Ioan Ciordas

CROATIA: High school Novska

GREECE: Geniko Lykeio Proastiou Karditsas

FRANCE: L'association Les Maths En Scène

## ● Neotrie VR ●



NeoTrie VR is a multiplayer virtual reality software, that allows users to create, manipulate and interact with geometric objects and 3D models in general, of different types, in a collaborative way.

The aims of the use of this software are:

- Plane geometry visible through the eyes of a third dimension;
- 3D geometry and modelling meant for 3D printing;
- Develops handicrafts and 3D visual skills;
- Stimulates deductive and inductive reasoning skills;
- Fosters cooperative work and positive interdependence;
- Motivates pupils by means of recreational, collaborative and competitive games.

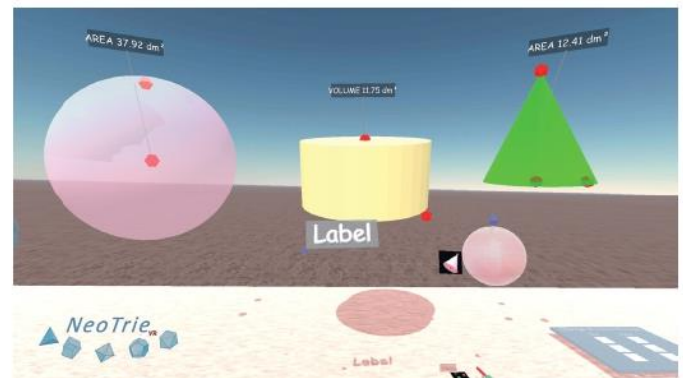
The software implements learning activities for different ages, covering a wide variety of topics.



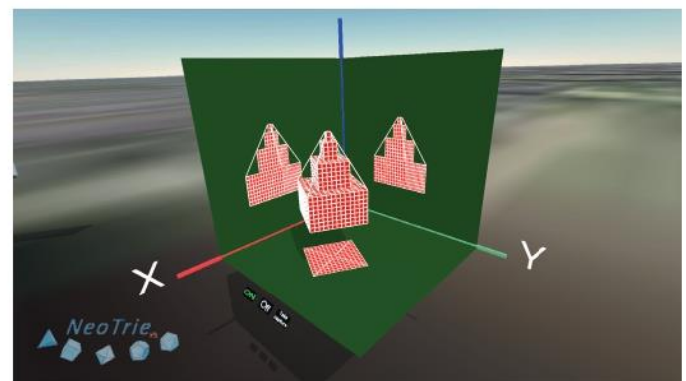
The standard tools of dynamic 3D geometry can be operated intuitively in the VR environment:



- 2D and 3D geometry (parallels, perpendiculars, bisectors,...) and metric calculations (lengths, angles, areas, volume,...).



- Projection of 3D figures in multiview planes.



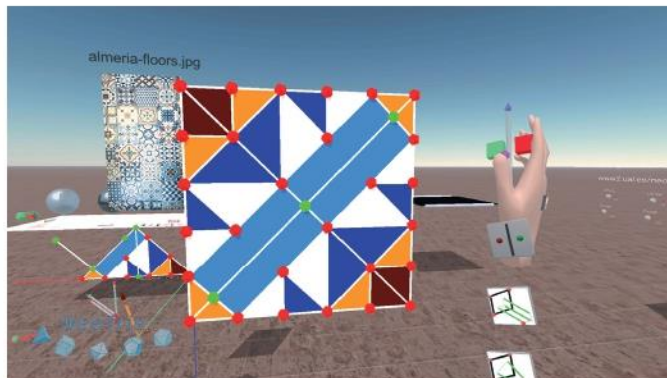


# Geometricians View's

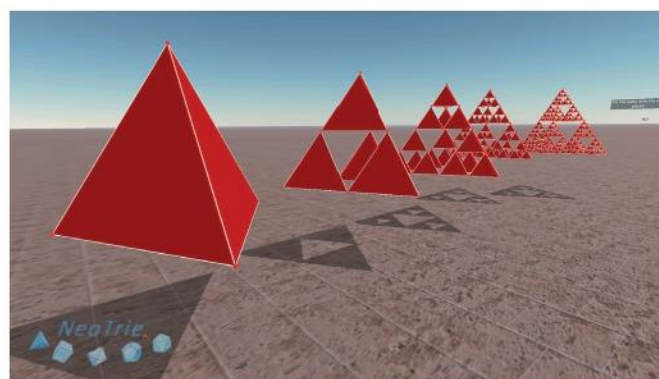
- Construction of famous polyhedra (Platonic bodies, pyramids, prisms, antiprisms, Archimedean, Johnson, Kepler, geodesic spheres, etc.), duality, truncation, rectification, extension,....



- Spatial symmetries to build 2d and 3d tessellations (translations, rotations, reflections,...).



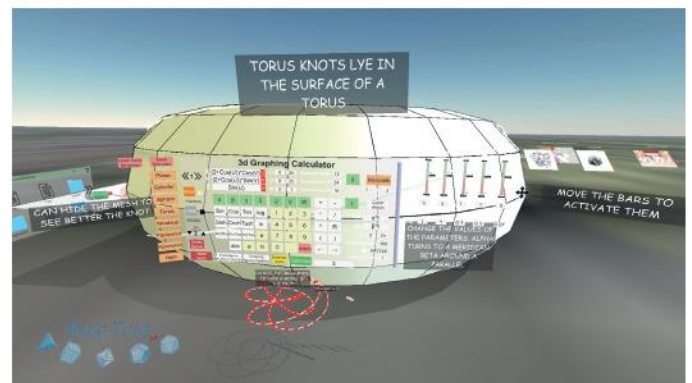
- Homotheties to build self similar fractals.



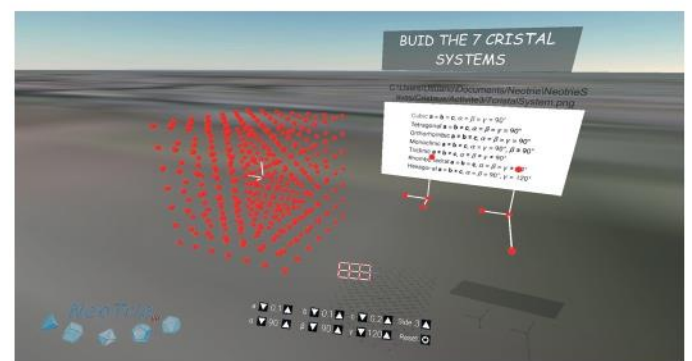
- Manipulate curved bodies, find intersecting curves.



- 3d graphing calculator for parametrized curves and surfaces.



- Crystal networks from fundamental cells.



- Eulerian paths, Hamiltonian paths on graphs, colouring graphs.



# Geometricians View's

## Features of Neotrie VR:

- Gallery of pre-designed geometric figures.
- File system to load and save activities created by users online.
- Photo camera inside the VR scene.
- Photos, videos, texts, sounds can be inserted in the scene.
- Exporting-importing from other 3D geometric softwares as STL objects.
- Importing GeoGebra files (alpha version).
- 3D printing from STL files edited in NeoTrie.
- Multiplatform: compatible with Meta, Steam VR, Vive, Valve Index, Pico4, etc.

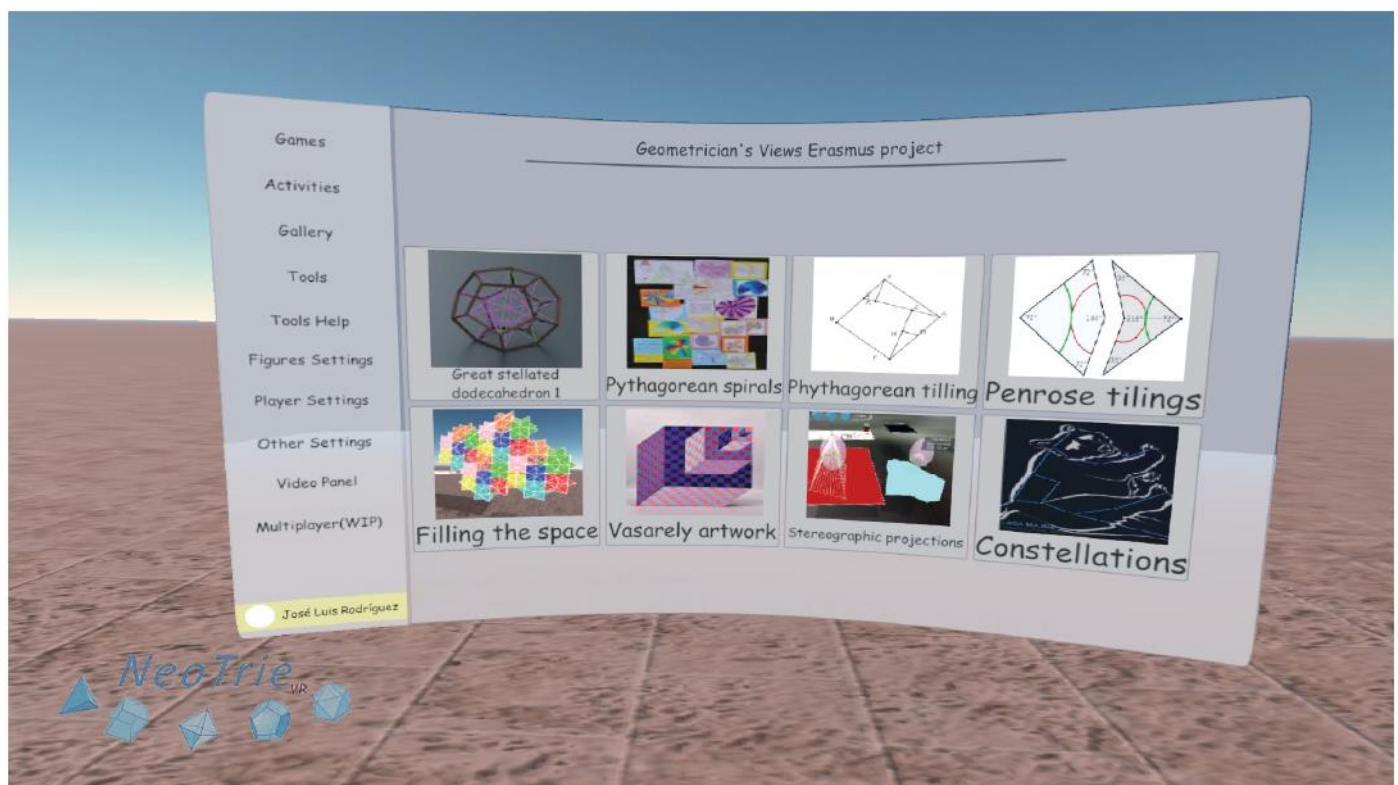
## Interactiv Virtual Museum

In this brochure you will find extra activities in Neotrie VR, which are accessible in the game by touching this cube in the scene.



Visit the following web page for detailed information on how to implement Neotrie VR in the classroom, and the complete list of activities:

<https://www2.ual.es/neotrie/project/geomview/>



A sample of the activities



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## ● Tutorials ●

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# MUSIC

**Mathematics, Nature, Science, Art = Beauty**



## Music and Maths

*"There is geometry in the humming of the strings,  
there is music in the spacing of the spheres."*

Pythagoras

The notations of composers and sounds made by musicians are connected to mathematics (see reference 1).

Music and mathematics have always been "friends"! Does it seem possible to you? Yet, it is not a new concept at all! The link between music and mathematics has already been discovered in ancient times and dates back to the genius of **Pythagoras**. He was the first to guess the existence of **numerical relationships between frequencies**. Through these, he built the first musical scale. But this relationship was then studied by many scientists, philosophers, and musicians such as Tolomeo, Galileo Galilei, Leibniz, and many others.

*"Music is the pleasure that the human mind feels when it counts without being conscious of counting" - Gottfried Wilhelm von Leibniz*

**1**

### EDUCATIONAL AIMS

Exploring the connections between **mathematics** and **music**

**2**

### TARGET PUBLIC

Pupils aged **12-14**  
Lower secondary school

**3**

### SKILLS NEEDED

Basic knowledge of physics, arithmetics, informatics, and drawing

**4**

### MATHEMATICAL AND ARTISTIC AIM

Connections between mathematical and musical **notations**

## Learning activities

### Learning activity 1: Making a monochord

Pupils age: 12-14 years old

Material: a wooden pole 65-70 cm long and 8 cm wide, two resistant pieces of wood, parallelepiped shaped, two movable bridges with a triangular section, two eyelets to screw, four strings for classical guitar (low E, A, D, G), meter, pencil, digital tuner (or a corresponding free app); plectrum for plucking the string.

Activity's detail: students will experimentally discover the working principle of a monochord and relationships between string lengths and sounds, applying mathematical concepts like the multiplication table, least common multiples, and prime and composite numbers.

Students' work: our monochord was built by Prof. Oreste Rinaldi and used during the laboratory activity guided by Prof. Simone Pacetti and Vladimiro Vagnetti.

### Learning activity 2: Mersenne's laws and acoustic instruments exhibition.

Pupils age: 12-14 years old

Material: squared notebook, pencil, calculator, internet.

Activity's detail: students will solve exercises inspired by Mersenne's laws guided by the researcher Simone Pacetti and explore acoustic instruments belonging to the Physics and Geology Department of Perugia University.

Students' work: [here is the tutorial of the students' laboratory.](#)

Activity 1 and 2: [Here are pictures of the laboratory.](#)

### Learning activity 3: Composing music with the Cartesian plane (Vladimiro Vagnetti)

Pupils age: 12-14 years old

Material: prepared cartesian sheets, colored pencils, rulers, teaching instruments (percussion instruments, xylophone, sound plates, boomwhackers, recorder) or classical instruments played by students, tablet or notebook.

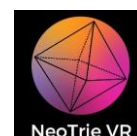
Activity's detail: concepts of pitch and duration, the first two parameters of sound, and their linear development in melody as a succession of pitches and rhythm as a succession of musical values; drawing a musical scale using the equation of a straight line ( $y = mx$ ); transforming a melody written in traditional notation into a graphic on the Cartesian plane and play the graph; composing a melody with Chrome Music Lab's Song Maker.

Students' work: [here is the tutorial of the students' laboratory.](#)

Activity 3: [Here are pictures of the laboratory.](#)

### Extra activity in Neotrie VR

<https://www2.ual.es/neotrie/project/geomview/#Music and geometry>



## References

1. [Mathematics & Music](#) - American Mathematical Society
2. [Mathematics of Music](#)

Image 1: Musical instruments drawn by Simone Pacetti and [Pythagoras](#) in the background.

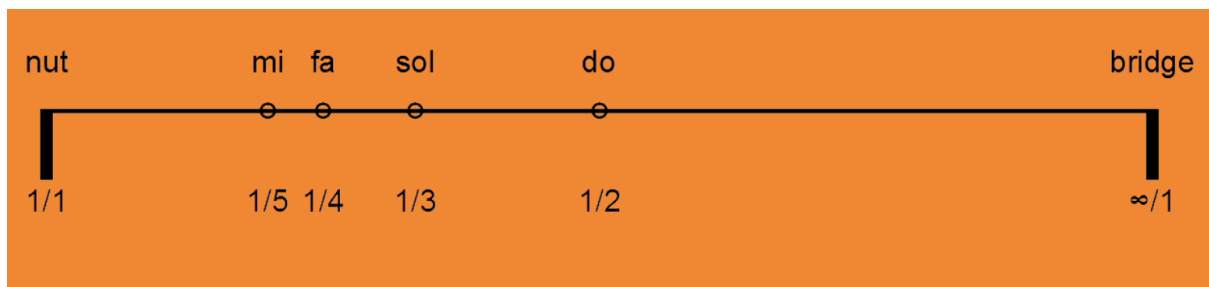
Image 2: [Monochord](#)

## LINKS

[Learning activity 1: Monochord](#)

[Learning activity 2: Mersenne's Laws](#)

[Learning activity 3: Song Maker](#)

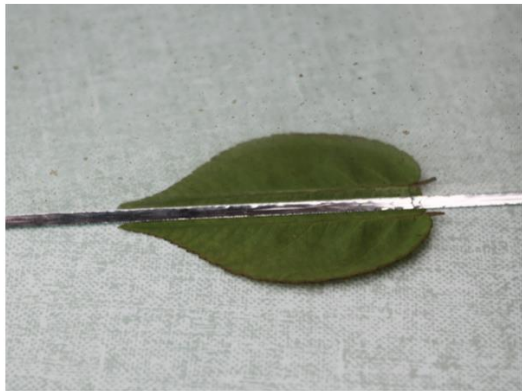






# MIRRORS

Mathematics, Nature, Science, Art = Beauty



## Mirrors and Maths

"In mathematics, reflection symmetry, line symmetry, **mirror symmetry**, or mirror-image symmetry is symmetry with respect to a reflection." Reference 1 (Wiki)

Mirror symmetry is also a fascinating topic in the field of **theoretical physics**. Students can therefore carry out simple reflection experiments, thinking that the symmetry concept is fundamental in science.

**Reflection** on a **mirror** is an easy activity to perform with students extremely rich in mathematical meaning. The **symmetry** concept that comes naturally out of a mirror reflection can be easily applied in **geometry**, **science**, and **art**. Looking for symmetry axes in geometrical shapes, living organisms, and architectural structures of buildings is a valuable educational activity for middle school students.

*"The importance of mirror-reflection symmetry to our perception and aesthetic appreciation, to the mathematical theory of symmetries, to the laws of physics, and to science in general, cannot be overemphasized, and I will return to it several times" - Mario Livio*

1

### EDUCATIONAL AIMS

Exploring the connections between **mathematics** and **mirror symmetry**

2

### TARGET PUBLIC

Pupils aged **12-14**  
Lower secondary school

3

### SKILLS NEEDED

Basic knowledge of physics, geometry, informatics, and drawing

4

### MATHEMATICAL AND ARTISTIC AIM

Exploring the **aesthetic value** of **symmetry** in **art**

## Learning activities

### Learning activity 1: Images reflections

Pupils age: 12-14 years old

Material: mirrors, paper, pencils, symmetrical shapes, and several drawings.

Activity's detail: students will start drawing polygons in order to identify and study the definition of a symmetry axis. Then, they will 1) draw half of the symmetric shapes; 2) look for symmetrical shapes in magazines and cut images; 3) find natural structures (like leaves) showing symmetry and cut along the symmetry axis. All images, natural structures, and drawings will then be used to identify the symmetry axis using a mirror. Eventually, they will geometrically verify the symmetry.

Students' work: [here is the tutorial of the students' laboratory.](#)

#### [Axis of Symmetry](#)

### Learning activity 2: Hinged mirrors (built by the teacher Oreste Rinaldi) and multiple images

Pupils age: 12-14 years old

Material: hinged double mirror, protractor, paper sheets, pencil, calculator.

Activity's detail: students will experimentally verify the formula  $n = 360^\circ/\alpha$ , where  $n$  is the number of images and  $\alpha$  is the amplitude of the angle. They will therefore think about how the number of reflected images changes in order to find out the relationship between the amplitude of the angle and the number of reflected images. They will also experience the difference between two situations: 1)  $\alpha$  is a factor of ; 2)  $\alpha$  is not a factor of .

Students' work: [Here are pictures of the laboratory.](#)

### Learning activity 3: Art at the mirror: the kaleidoscope (guided by the teacher Laura Ceccarelli)

Pupils age: 12-14 years old

Material: colored cardboard, glue, small and colorful translucent beads, aluminum foil or silver reflective paper, pencil, scissors, clear tape, colors, and small clear plastic travel jars.

Activity's detail: the activity is aimed at constructing a hand-made kaleidoscope, colorful and very easy to use as explained in the learning activity link (3). Students will admire the beautiful patterns obtained and try to explain how they form.

Students' work: [Here are pictures of the laboratory.](#)

## References

1. [Reflection symmetry](#)
2. [Mirror Symmetry \(String Theory\)](#)
3. [Mathematicians Explore Mirror Link Between Two Geometric Worlds](#)

Image 1: Reflection of a leaf by Oreste Rinaldi

Image 2: [Cross section of the quintic Calabi–Yau manifold](#)

## LINKS

[Learning activity 1: Symmetry](#)

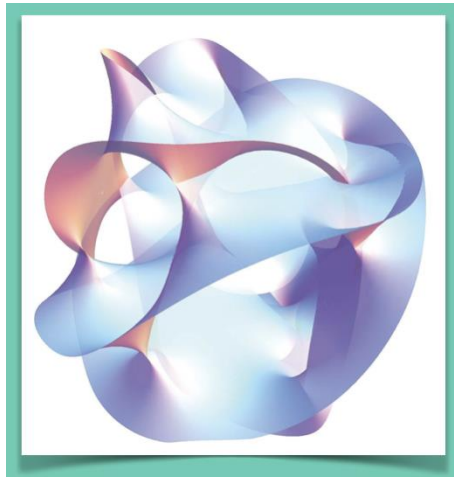
[Learning activity 2:](#)

[Multiple Images](#)

[Right Angle Mirrors](#)

[Other Multiple Mirror Systems](#)

[Learning activity 3: How to Build a Kaleidoscope](#)



### Extra activity in Neotrie VR

Students play in multiplayer mode with symmetries in virtual reality, with the parallel, mirror and rotation tools.



<https://www2.ual.es/neotrie/project/geomview/#Symmetries>





## Once upon a time there were paving stones...

### 1. Introduction

Paving stones offer a very rich palette of investigation, adaptable to all ages and all sensibilities.

Paving stones are part of our daily lives (tiles, wallpaper, decorative objects, games, etc.), thus providing direct observation material. They are also very present in the course of history, from the Roman paved roads, through the pavements of medieval Islamic art, and more recently the pavements of the mathematician Roger Penrose or the engraver Maurits Cornelis Escher

It is therefore a question of exploring all these riches, painting, architecture, everyday art... and of using them to teach mathematics in order to acquire and consolidate new knowledge.

### 2. Summary of the activity

- Educational aim
- To anchor geometry in reality, to acquire knowledge and skills useful for everyday life and even for professional life,
- To reinforce the ability to reason and argue,
- To learn to anticipate, plan tasks, manage the stages of a construction programme, a project.
- Target Age: 9/11 years

Competence required:

- Use geometric tools: ruler, square, compass
- Know the properties of simple figures
- Know how to construct simple or complex figures (assemblies of simple figures)

Mathematical and artistic aim :

- Create works and games containing tessellations

### 3. Learning activities

Each of the activities should be understandable by a teacher, so give as much detail as possible. For each activity, see if Neotrie can propose something. For each of the activities indicate :



### **activity 1: Regular tiling game**

Age: 9/11 years

Method of work: Individually, then in groups

Materials: cardboard to create figures, geometry tools (compass, ruler, square), cardboard model or plywood board for the game pieces, digital tool for creating the challenge cards

Objective: to discover tessellations, to make regular polygons (square, equilateral triangle, hexagon), to manipulate to create various patterns, to create pattern cards using the digital tool for the game

Course of the activity:

<https://digipad.app/p/148070/7b96433bd6a85>

Student work:

<https://digipad.app/p/148070/7b96433bd6a85>



### **activity 2: Game "Truchet's challenges**

Age: 9/11 years

Method of work: Individually, then in groups

Materials: cardboard to create the two-coloured tiles, geometry tools (compass, ruler, square), cardboard model for the game pieces, felt pens,

Objective: make two-coloured squares, manipulate to create various patterns using translation, rotation, symmetry, create challenge cards for the game

Course of the activity:

<https://digipad.app/p/148070/7b96433bd6a85>

Student work:

<https://digipad.app/p/148070/7b96433bd6a85>



### **Activity 3: Fresco "Zelliges**

Age: 9/11 years

Working method: Individual and then in groups

Materials: cardboard sheet to create zellige templates, black, blue, white and red canson sheet, geometry tools (compass, ruler, square), felt pens, digital tool to write the quote

Objective: To discover a new culture, to follow a construction programme to create a new motif (zellige) using the equilateral triangle as a basic figure, to create a fresco by paving a space with zellige,

Course of the activity:

<https://digipad.app/p/148070/7b96433bd6a85>

Student work:

<https://digipad.app/p/148070/7b96433bd6a85>



#### **Activity 4: Paving, weaving...**

Age: 9/11 years

Method of work: Individual and then group work

Materials: old newspapers, a stapler, transparent self-adhesive paper (like book covers)

Objective: To make a checkerboard using the weaving technique, to cooperate to make a collective work

Course of the activity:

<https://digipad.app/p/148070/7b96433bd6a85>

Student work:

<https://digipad.app/p/148070/7b96433bd6a85>

## **4. Documents used**

Papiers Crayons is a collection of activities for drawing on a grid and building with a ruler and compass, published by the IREM of Paris-Nord.

Geometry for pleasure. Jocelyne and Lysiane Denière

## **5. Webography**

<https://publimath.univ-irem.fr/glossaire/PA041.htm>

[https://www.apmep.fr/IMG/pdf/I\\_Pavages\\_du\\_plan\\_ESCHER.pdf](https://www.apmep.fr/IMG/pdf/I_Pavages_du_plan_ESCHER.pdf)

[https://www.apmep.fr/IMG/pdf/3\\_diapo\\_alhambra-12\\_mars.pdf](https://www.apmep.fr/IMG/pdf/3_diapo_alhambra-12_mars.pdf)





# SPIRALS

A spiral is a curve that starts at a point and moves farther and farther away as it rotates around that point.



**The study of spirals in nature** has a long history. Christopher Wren observed that many shells form a logarithmic spiral, Jan Swammerdam observed the common mathematical characteristics of a wide range of shells from Helix to Spirula, and Henry Nottidge Moseley described the mathematics of univalve shells. D'Arcy Wentworth Thompson offers an extensive study of these spirals. He describes the way in which shells are formed by rotating a closed curve around a fixed axis: the shape of the curve remains fixed, but its size increases in a geometric progression. In some shells, such as Nautilus and ammonium, the generating curve rotates in a plan perpendicular to the axis, and the shell will form a discoid-flat shape. Thompson also studied spirals that appear in horns, teeth, claws, and plants.

Spirals in plants and animals are often described as whirlpools. This is also the name given to spiral fingerprints.

## Gold Section

### Short history. Theoretical support.

The gold section (also called the gold ratio, gold proportion, or gold number), denoted by the Greek letter, is the first irrational number discovered and defined in history. It is

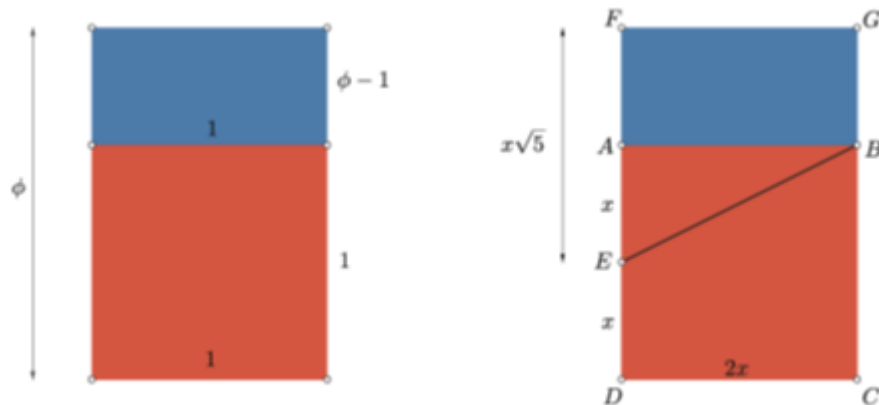
approximately equal to 1,618033 and can be encountered in the most surprising circumstances.

The gold section of the length segment  $a+b$  is made when the ratio between  $a+b$  and  $a$  is equal to the ratio between  $a$  and  $b$ . In this illustration  $a$  it is called "extreme ration" and  $b$  is called "average". Euclid called as the simple division of a right segment into what he called "average" and "extreme ration." Here are his words: "We say that a segment of the right has been divided into average and extreme ration when the whole segment relates to the larger segment as it relates the largest segment to the smallest."

In other words, if  $\frac{a+b}{a} = \frac{a}{b}$  then the segment  $a+b$  was divided into a golden section with the symbol  $\varphi$ .

The golden ratio is an irrational number that can be calculated from the relationship  $\frac{a+b}{a} = \frac{a}{b} = \varphi$  which leads to the equation  $\varphi^2 - \varphi - 1 = 0$  that means  $\varphi = \frac{\sqrt{5}+1}{2}$ .

We consider a rectangle with the ratio of the lengths of the sides  $1/\varphi$ . We determine the number  $\varphi$  so that, if we divide the considered rectangle into a square and a rectangle, the ratio of the lengths of the sides of the obtained rectangle is also  $1/\varphi$ . Such a rectangle is called "**golden rectangle**" (the ratio of the sides is equal to the golden number „ $\varphi$ ”).



To calculate the value of the number  $\varphi$  we notice that

$$\frac{\varphi}{1} = \frac{1}{\varphi-1} \Leftrightarrow \varphi^2 - \varphi - 1 = 0,$$

where

$$\varphi = \frac{\sqrt{5}+1}{2}.$$

Euclid used the following construction to obtain the golden rectangle: In the square  $ABCD$  of the side if  $E$  is the middle of the side  $AD$  then

$$EB = \sqrt{x^2 + 4x^2} = x\sqrt{5}.$$

We consider now the point  $F$  on the semi-right (DA, so that

$$EB = EF = x\sqrt{5}$$

and we build the rectangle FGCD. As

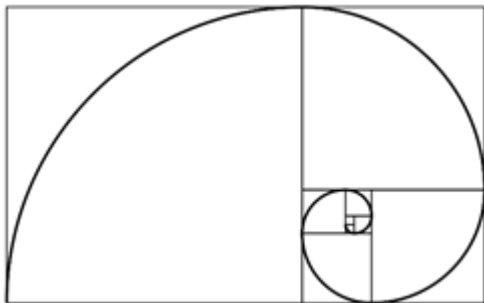
$$\frac{FD}{CD} = \frac{x\sqrt{5} + x}{2x} = \frac{\sqrt{5} + 1}{2} \quad \text{si} \quad \frac{FG}{AF} = \frac{2x}{x\sqrt{5} - x} = \frac{2}{\sqrt{5} - 1} = \frac{\sqrt{5} + 1}{2},$$

the rectangle wanted one.

## Logarithmic Spiral

### Theoretical support

The successive points that divide a golden rectangle into squares are on a logarithmic spiral, which is also known as the gold spiral.



The logarithmic spiral has the polar equation

$$r = a \cdot e^{b\theta},$$

where  $r$  is the distance from the origin, and

$\theta$

is the angle formed by the axis and  $a$  and  $b$  are two arbitrary constants.

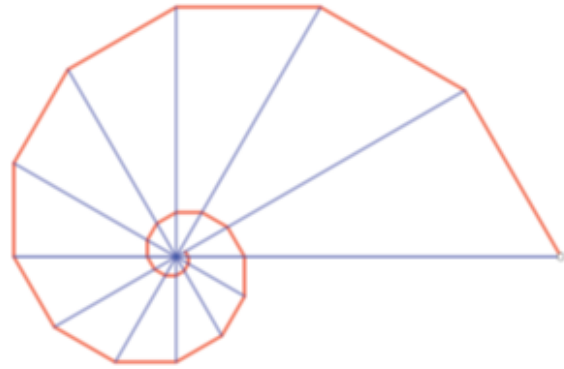
The parametric equations of the

logarithmic spiral are

$$\begin{cases} x = r \cos \theta = a \cdot e^{b\theta} \cos \theta \\ y = r \sin \theta = a \cdot e^{b\theta} \sin \theta \end{cases}$$

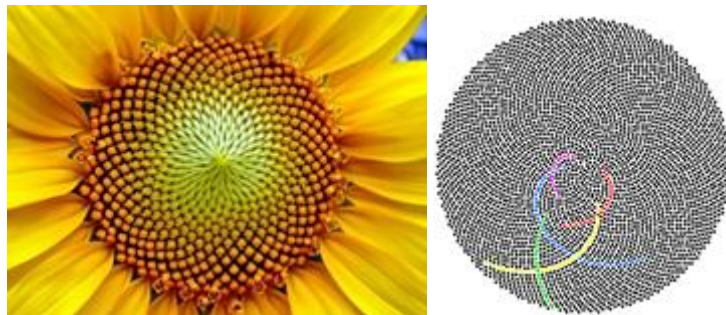


This spiral is related to the Fibonacci numbers, the golden number and the golden rectangle, sometimes called the golden spiral. The logarithmic spiral can be constructed from equally spaced radii, starting from a point on one of the radii and drawing perpendicular to a neighbouring radius. As the number of rays increases (approaches infinity), the sequence of segments approaches the smooth logarithmic spiral.



The logarithmic spiral was first studied by Descartes in 1638 and by Jakob Bernoulli. Bernoulli was so fascinated by the spiral that he had one engraved on his tombstone with the words *eadem mutata resurgo* (I will appear the same, though changed). Torricelli worked independently and found the length of the curve.

The logarithmic spiral is frequently found in nature, for example in the arrangement of flower seeds. For instance, it is easy to see that the sunflower seeds are arranged in such a way that they form two series of logarithmic spirals: one series of curved spirals clockwise and the other series of curved spirals counterclockwise.



Moreover, at the sunflower, the two directions of curvature of a specimen do not have an equal number of spirals, but two successive numbers in the Fibonacci sequence: In the normal, medium flower (with a diameter of about 12-15 cm) you can usually count 34 spirals in one direction and 55 spirals in the other direction. To small specimens 13 and 21 or 21 and 34 spirals and to the large specimens up to 89 and 144 spirals. The logarithmic spiral is also found in the profile of many **shell species**.



**Astronomers** were surprised to discover logarithmic spirals on their photographic plates: this is the shape of spiral nebulae and the tails of most comets.

Visual artists and theorists, who have studied the **proportions of animal bodies** and especially **the proportions of the human body**, have found that the gold section is the basic geometric canon of these proportions. Leonardo da Vinci and Albrecht Durer established that the navel divides the height of the human body by the golden section and observed that the other elements of the human body are subordinated to the proportion of gold.



#### **At the snail:**

-As the snail develops, it needs, repeatedly, a larger space. Since the snail does not change its shape, but only its size, the most effective way to accommodate it is to develop the shell in the shape of a logarithmic spiral.



#### **At the hawk:**

-The hawk must always have prey in its field of vision. Although it has an excellent view, the position of the eyes is fixed and lateral. What the hawk does is to incline its head at a 40-degree angle and look at its prey with one eye. Keeping his head tilted at 40 degrees, the hawk attacks its prey without losing sight of it. The fixed angle of his head is the result of the spiral with sharp angles.

<b>EDUCATIONAL AIMS</b>	<b>TARGET PUBLIC</b>	<b>NECESSARY ABILITIES</b>	<b>MATHEMATICAL AND ARTISTIC COMPETENCES</b>
Students know the proportion of gold, the golden rectangle as concepts that emphasize the harmony of the environment.	Pupils aged between 12 and 16 years old.	Basic knowledge of geometry, drawing and ICT.	<p>Students will identify the presence of proportions and mathematical concepts in the environment.</p> <p>Students will have the opportunity to transfer knowledge and scientific notions in everyday life admiring what surrounds them and being creative in making the final product.</p>

## LEARNING ACTIVITIES

### Learning activity 1: Theoretical notions about the golden number (golden section), the golden rectangle

**Age of students:** 12-16 years

**Material resources:** computer, internet connection, coloured pencils, geometric tools, flipchart, Geogebra application, YouTube documentaries

#### Activity details:

- The theoretical notions are presented to the students.
- Students will watch the documentaries suggested by the teacher, accessible from the internet

<https://www.youtube.com/watch?v=4XN8LSfrEpU>

Students will be divided into groups of 3 and will answer the following questions: **1.** What is the golden number and what is its approximate value? **2.** How is the golden number also called? **3.** What is the golden rectangle? For question **4** students will look for information about the golden rectangle and describe examples for the following categories: nature, human body, architecture, art. (One category at a time, chosen by drawing lots). **5.** A golden rectangle will be built following an algorithm similar to the one discovered by Euclid. A team leader will present to the class the information selected and written on the flipchart.

For requirement **no. 5** the teacher gives the explanations.

The golden rectangle is part of the series of dynamic rectangles that can be built with the ruler and the compass, using the rotations of some segments, meaning continuous movements, producing a variety of subdivisions.

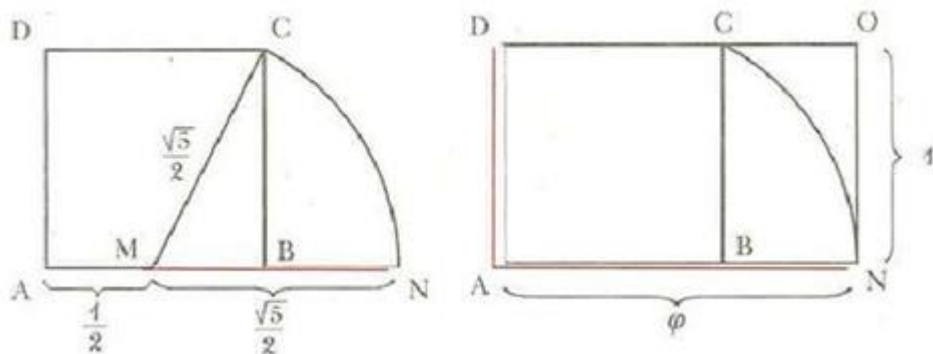
The ratio of the sides of a golden rectangle is:

$$\frac{L}{l} = \varphi.$$

The rectangle is constructed as follows: Starting from a segment AB which can be its length or width, the extension of the segment is constructed obtaining the point N, so that:  $AB/BN = AN/AB = \rho$ . Thus, we obtain that ANOD and BNOC are golden rectangles and the rectangle ABCD is a square.

We can obtain golden rectangles if we attach to a golden rectangle a square on the side of the length of the segment AB, just as a golden rectangle results if we construct a square on its width in a golden rectangle and look at the new rectangle formed which is a golden rectangle.





**Note.** In case due to the sanitary measures the distance must be kept, the students will cooperate on the Teams platform, keeping the division into groups. The writing on the flipchart will be replaced with the design of the final product. The final products will be analyzed from a scientific and artistic point of view, and will be photographed.

#### **Bibliography/ resources:**

[https://www.academia.edu/35385107/Dreptunghiul\\_de\\_aur](https://www.academia.edu/35385107/Dreptunghiul_de_aur)

<https://crash4crantz.wordpress.com/tag/dreptunghiul-de-aur/>

<http://vasileteodor.ro/articol/numerele-lui-fibonacci-si-proportia-de-aur>

#### **Learning activity 2: Theoretical notions about the logarithmic spiral**

**Age of students:** 12-16 years old

**Material resources:** computer, internet connection, coloured pencils, geometric tools, flipchart, Geogebra application, YouTube documentaries

#### **Activity details:**

- The students are presented the theoretical notions about the logarithmic spiral and its drawing algorithm.

- Each student draws on a board a logarithmic spiral starting from a different number of rays (depending on the number extracted, between 12 and 18).

- Each student makes a product that represents a logarithmic spiral, using the desired technique.

*The final products will be analyzed from a scientific and artistic point of view, and will be photographed.*

#### **Bibliography/ resources:**

<https://iulianandrei.wordpress.com/2011/01/11/spectacolul-spiralei-si-efectele-sale/>

<https://demonstrations.wolfram.com/Spirals/>

### Extra activity in Neotrie VR:

Students can build the pythagorean, logarithm and archimedean spirals in 3d space.



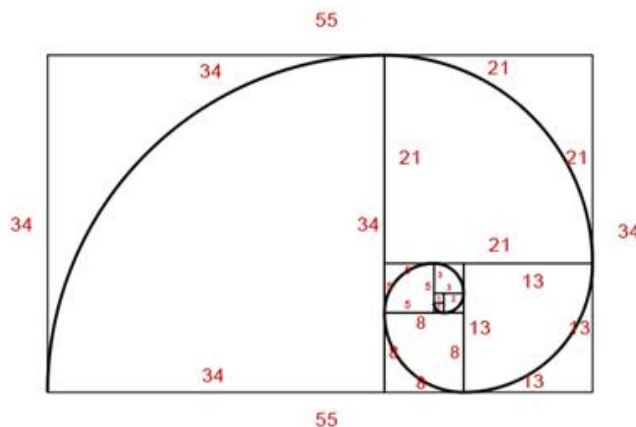
<https://www2.ual.es/neotrie/project/geomview/#Spirals>

### In conclusion:

**Have you noticed how much symmetry and order there is in nature?** People are constantly trying to understand nature and its laws, to feel the cosmic rhythms, to actually understand life more deeply, in order to reach a harmony with the environment.

**The applications of the golden number**, in fact of the ratio as such, are found in the proportion of works in architecture, painting, sculpture, aesthetics and art in general, which confirms the interest shown over time for this number. The proportion of gold (divine) led to the construction of the "golden rectangle", in which the ratio of the sides is equal to the golden number. This type of rectangle is considered to be particularly aesthetic and, as a result, has been and is intensely used in architecture and art.

As a geometric application of the Fibonacci sequence, golden rectangles are



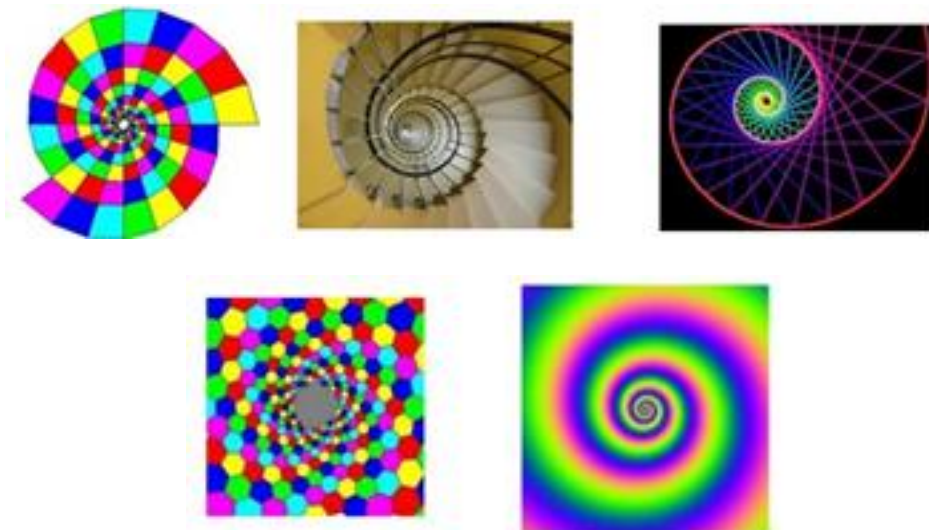
Șirul lui Fibonacci: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946...

constructed:

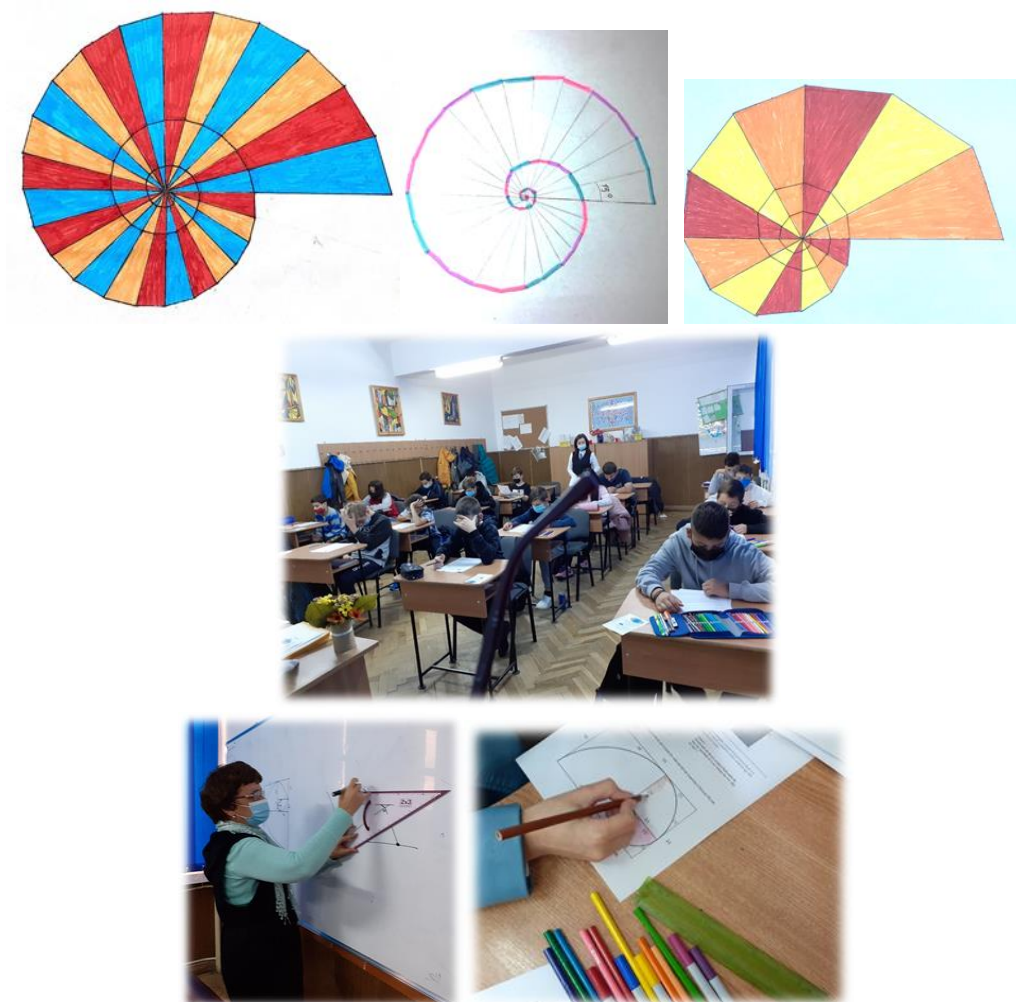
Among the plane curves researched in mathematics, algebraic spirals (Archimedes' spiral, hyperbolic, etc.) and pseudospirals (logarithmic, circle evolution, etc.) are highlighted. Spirals have a wide application in technology, the machine building industry, telecommunications and in the execution of construction and geodetic works.

## BIBLIOGRAPHY

- 1) Robert W. Fathauer, Logarithmic Spiral Tilings of Triangles, Bridges 2021 Conference Proceedings, <https://archive.bridgesmathart.org/2021/bridges2021-55.pdf>
- 2) <https://en.wikipedia.org/wiki/Spiral>
- 3) [https://ro.wikipedia.org/wiki/Secțiunea\\_de\\_aur](https://ro.wikipedia.org/wiki/Secțiunea_de_aur)
- 4) <https://mathworld.wolfram.com/GoldenRatio.html>
- 5) <https://mathworld.wolfram.com/GoldenRectangle.html>
- 6) [https://math.wikia.org/ro/wiki/Spiral%C4%83\\_logaritmic%C4%83](https://math.wikia.org/ro/wiki/Spiral%C4%83_logaritmic%C4%83)
- 7) <https://mathworld.wolfram.com/LogarithmicSpiral.html>
- 8) The theoretical support made available by Assoc.Dr. Fechete Ioan, University from Oradea, Faculty of Mathematics and Sciences.
- 9) [https://www.youtube.com/watch?v=\\_273\\_rnf2nA](https://www.youtube.com/watch?v=_273_rnf2nA)
- 10) <https://www.youtube.com/watch?v=MMhqqQf68Jg>



## Photo snapshots during class activities







# PUZZLES

The word **puzzles** refers to a type of game, toy or problem that requires ingenuity, logic, vision and perseverance to reach a solution. It is a word originally from the English language, in a loose translation meaning decoupage.

The puzzle game consists in reconstructing an image or a three-dimensional object with numerous pieces (different in shape and colour) that can be interspersed.

The history of this type of game starts from a painting or a drawing made on a wooden board which is then cut into small shapes that can then be reassembled. After about 300 years since the first forms of the game, today the puzzle is made of cardboard, as this material is more convenient. Also, instead of painting, printing intervenes, so games can have a high content of details as well as greater freedom in choosing the image to be created. After the pattern, the cardboard is cut out using a metal die-knife following a complex contour.

The game can have from 4-9 pieces to thousands of pieces, thus increasing the difficulty but also the interest of the enthusiasts.

The application <http://www.jigzone.com> comes to our aid in order to fragment an image into several pieces to be assembled to reconstruct the original image.

Math puzzles do not require advanced math knowledge.

Unlike mathematical games, part of recreational mathematics, which involve the training of two or more players, we can say that puzzles can be solved by one or more players.

The field of recreational mathematics stimulates curiosity and inspires the desire to deepen mathematics.

Among the most famous math puzzles are:

### **Tangram**

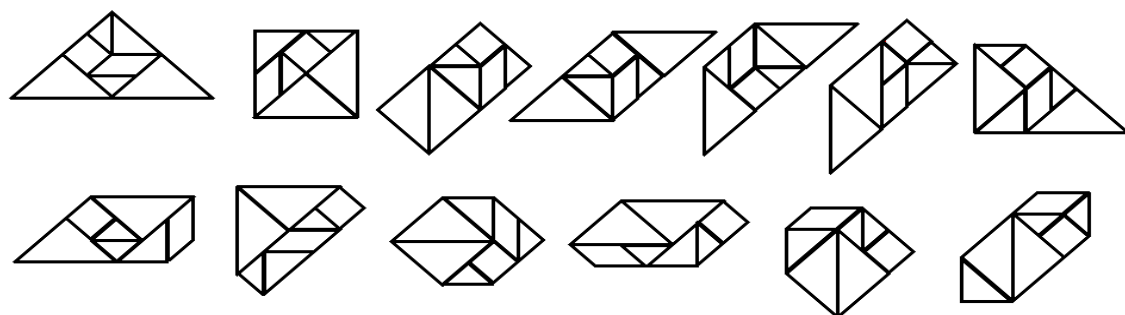


Tangram is a game invented by the Chinese, designed to be played by one person. The initial resources of the game are seven geometric figures (5 triangles (of different sizes), a square and a parallelogram).

The game consists in placing the 7 figures (all and only them - the first rule), side by side, without overlapping (the second rule), in the plane (the third rule), to form certain given figures, figures with aesthetic value or stylized images of real objects.

The 7 initial figures - they are also called tans - come from cutting a square in a certain way, as in the figure. They were traditionally made of stone, bone, clay or other "classic" materials, but today they can be made of plastic, wood, or other "modern" materials.

Related to Tangram, the question was asked: How many convex figures can be made with the help of the 7 pieces? One is the starting square, 12 others are shown in the adjacent figure.



In 1942 it was shown that only these convex figures can be made using the rules mentioned above.

Other math puzzles are considered:

The SOMA Cube

The Rubik's Cube

Rubik's Domino

Solitaire

Polyominoes

Sudoku

---

[1] <https://ro.wikipedia.org/wiki/>

EDUCATIONAL GOALS	TARGET AUDIENCE	REQUIRED SKILLS	MATHEMATICAL AND ARTISTIC SKILLS
Students should be stimulated to be creative, to be encouraged to think logically, to be rigorous, necessary qualities in the study of mathematics	Students between the ages of 12 and 18	Basic knowledge of geometry, drawing and ICT	-students will train their thinking by solving some puzzles;  -students will have the opportunity to be creative by making figures using the tangram pieces.

### LEARNING ACTIVITIES

**Learning activity 1:** Theoretical notions about puzzles, as a mathematical game that trains thinking and logic, stimulates creativity using the online resource at:

[https://en.wikipedia.org/wiki/Recreational\\_mathematics](https://en.wikipedia.org/wiki/Recreational_mathematics)

**Age of students:** 12-18 years

Material resources: computer with internet connection, the theoretical material above, online documentation

**Activity details:**

- Students answer the question "Have you heard of recreational mathematics? If so, where did you hear about recreational/fun math? Give examples of math games."
- Students, divided into groups, study the theoretical material presented above.
- A student presents the rules for completing a "Sudoku" square. Each team is given a "Sudoku" box to complete. The winning team is awarded.

- As homework: Students are given a cardboard square that they have to cut out to have the Tangram pieces.

## **Learning Activity 2: Creativity and Imagination**

Age of students: 12-18 years

Material resources: computer, internet connection, Tangram pieces

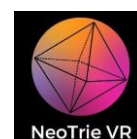
Activity details:

- Students will search the Internet for models of figures made with Tangram pieces and proceed to make similar figures.
- Within the groups, they will describe the Tangram pieces referring also to their dimensions if starting from a square of length "L".

## **Extra activity in Neotrie VR**

With the help of the NeoTrie software, certain figures will be made, using figures identical to the Tangram pieces.

<https://www2.ual.es/neotrie/project/geomview/#Puzzles>



Conclusions: We all want young people/children/students to improve their math skills. As adults, we can help them to understand the mysteries of mathematics with the help of fun mathematical games. In this way, children will develop and learn mathematical skills very easily, they will become more rigorous, more patient and more competitive. One of our duties is to present them with this option to spend their free time solving puzzles.

## **Bibliography & webography/:**

- [https://en.wikipedia.org/wiki/Recreational\\_mathematics](https://en.wikipedia.org/wiki/Recreational_mathematics)
- <https://en.wikipedia.org/wiki/Tangram>
- <https://sudokus.ro/>
- <https://www.247sudoku.com/>
- <https://en.wikipedia.org/wiki/Sudoku>
- Material selected by Prof. Fechete Ioan - Oradea University

Note. This material is inspired by the site <https://en.wikipedia.org/>, being useful in the implementation of the „PUZZLES" theme within our project.





## Erasmus + Geometrician's view



## Mathematics, Painting, Programming, Art= Beauty

**Pixel art:** what does it mean ?

Pixel art is a form of digital art drawn with graphical software where images are built using pixels as the only building block. It was associated with the low-resolution graphics from 8-bit and 16-bit era computers. The art form is still employed to this day by pixel artists.



**Gerhard Richter 4900 Colours:**  
*Version II* Serpentine Gallery

The word pixel is a combination of pix (from "pictures", shortened to "pics") and el (for "element"). The word "pixel" was first published in 1965 by Frederic C. Billingsley of JPL, to describe the picture elements of scanned images from space probes to the Moon and Mars.

A pixel is generally thought of as the smallest single component of a digital image.

The number of distinct colours that can be represented by a pixel depends on the number of bits per pixel (bpp). A 1 bpp image uses 1 bit for each pixel, so each pixel can be either on or off. Each additional bit doubles the number of colours available, so a 2 bpp image can have 4 colours, and a 3 bpp image can have 8 colours:

1 bpp,  $2^1 = 2$  colours (monochrome) / 2 bpp,  $2^2 = 4$  colours / ... / 24 bpp,  $2^{24} = 16,777,216$  colours ("TrueColor")

Educational aims	Target public	Needed Skills	Mathematical and artistic aim
Developing programming skills in an artistic field. Creating pictures from famous abstract painting models.	Pupils aged 15-16 Secondary school	Basic knowledge of Python programming	Recognizing mathematical patterns and encouraging artistic creativity

## LEARNING ACTIVITIES

### Learning activity 1: Regular polygons

**Aim:** Create python codes using the turtle mode to display some regular polygons. Students have to learn how to move linearly and rotate a turtle with the turtle mode. They have to master knowledge about regular polygons, their sides and interior angles so that then can make right changes in the code to display more complex figures.

**Pupil age:** 15-16 years old

**Material:** Computer, Edupython or Pyzo Software, a dictionary with elementary instructions in Python may be provided, should students need it.

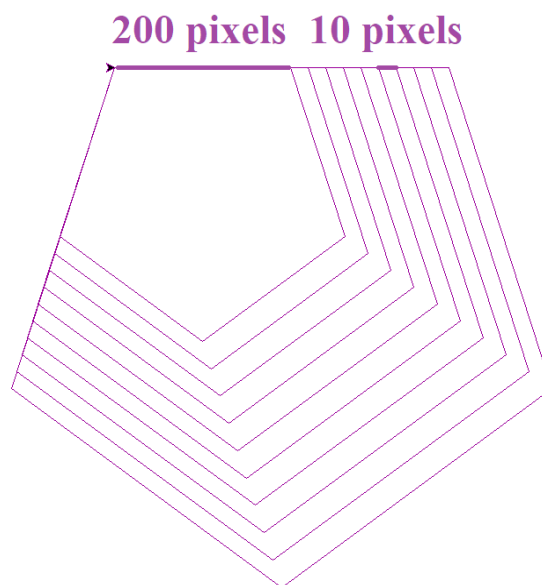
**Details:** The activity sets on five progressive questions for fully exploring both topics of regular polygons in geometry and turtle programming codes.

First question asks about a code to fill in so that the execution of the function can draw a square: the parameter is the length of each side.

Second question is related to the drawing of two combined squares whose lengths are known and different.

Third and fourth questions are about the drawing of regular hexagon and decagon. Students have to find out the link between the number of sides and the angle for each. The process is looped with a two instructions block (translation and rotation) that is repeated.

Last question is a challenge: students know some lengths and a drawing of combined pentagons they have to code ... They may correct themselves after each execution of their own code with a single comparison of both pictures: are they exactly similar or not ?



## Learning activity 2: Vasarely and the turtle

**Aim:** Create python codes using the turtle mode to display twelve complex figures. Those figures are progressive, students have to declare loop instructions and are involved to create subprograms, functions and combine them all. All figures have repetitive patterns based on regular polygons

In the second part, students are asked to create a picture close to a Vasarely's painting with a squared grid and coloured circles inside.

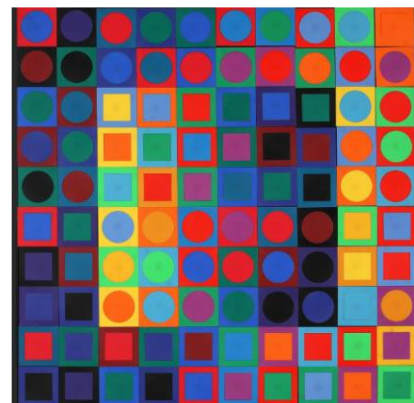
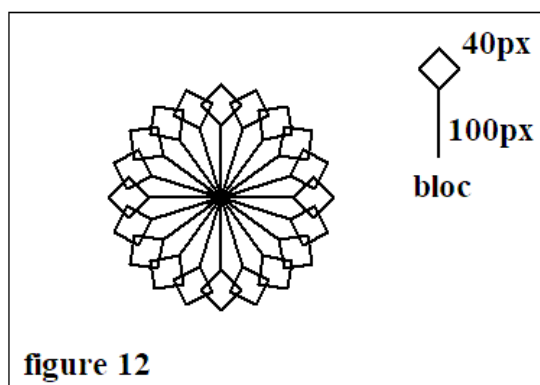
**Pupil age:** 15-16 years old

**Material:** Computer, Edupython or Pyzo Software, a dictionary with elementary instructions in Python may be provided, should students need it.

**Details:** The activity sets on five progressive questions for fully exploring topics of regular polygons in geometry, subprograms and turtle programming codes.

Similarity between pictures to realize and pictures displayed by execution of their codes allows a real self-control over coding activity.

Vasarely's painting is a creative and complex task. Students may choose the way to color all square, all circles and they have to think about subprograms and take care about the instructions order.



### Learning activity 3: Mondrian and the turtle

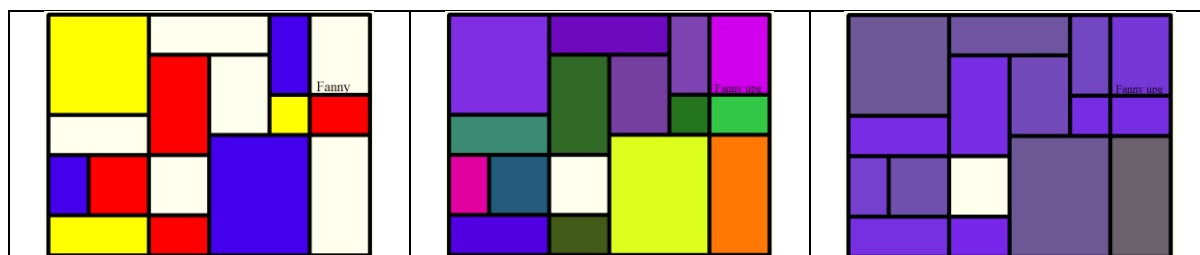
**Aim:** The aim is to create a python code using the turtle mode to display a picture looking like Mondrian's masterpiece *Composition II in Red, Blue, and Yellow*.

Students have first to analyse what is specific in Mondrian's composition, especially coloured rectangles as a pattern. They need to invest their comments to program a function "rectangle", to loop this function and add both random colours and moves that allow to talk about translations and rotations.

**Pupil age:** 15-16 years old

**Material:** Computer, Edupython or Pyzo Software, a grid can be provided if students want to begin imagining their picture on a sheet of paper.

**Details:** The teacher provides a document that helps students to program with the turtle mode. They have to think about the order of the instructions so that they can produce a picture that looks like Mondrian's and that fits in a same frame. The turtle mode colours a graphics inside a frame pixel by pixel. Students have to choose colours and so they work on RGB codes, 3-tuples of integers between 0 and 255 and, as a result, think about how many colours could be created. They also experience a random process to get a different picture for every execution. Different modes catch the eye: Mondrian's primary colours, random colours or a slight blue gradient.



For instance, here is Fanny's work: same frame and three modes.



## Links

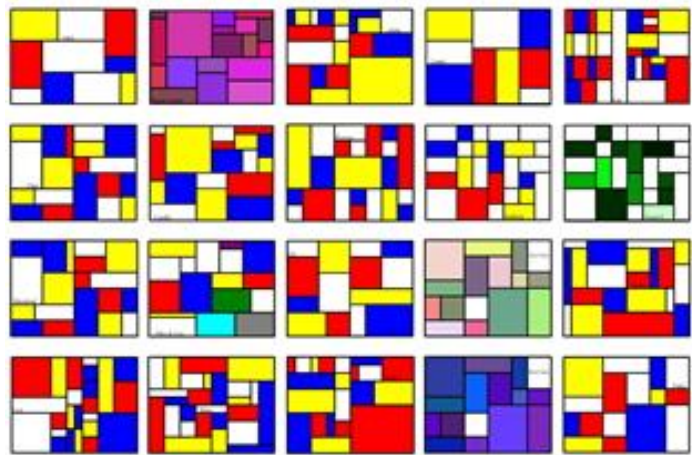
[All files in this folder](#)

Learning activity 1:  
Instructions sheet and programs

Learning activity 2:  
Instructions sheet and programs

Learning activity 3:  
Instructions sheet and programs

[Website report](#)



## References

- Python tutorial: <https://docs.python.org/3/tutorial/>

- Where to download Pyzo: <https://pyzo.org/start.html>

- About Pixel Art:

[https://en.wikipedia.org/wiki/Digital\\_painting](https://en.wikipedia.org/wiki/Digital_painting)

[https://en.wikipedia.org/wiki/Computer\\_art](https://en.wikipedia.org/wiki/Computer_art)

[https://en.wikipedia.org/wiki/Generative\\_art](https://en.wikipedia.org/wiki/Generative_art)

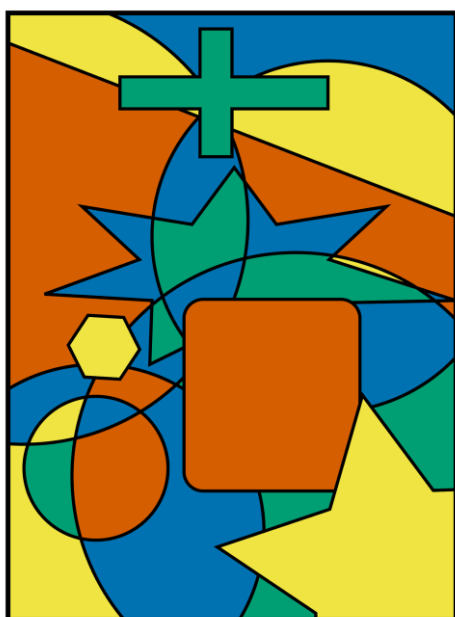
- About Vasarely's painting art: <https://www.artnet.com/artists/victor-vasarely/>

- About Mondrian's painting art: <https://www.artnet.com/artists/piet-mondrian/>



## COLORING

Coloring is the process or art of applying color.



[image source](#)

Color is the element of art that is produced when light, striking an object, is reflected to the eye.

But in art design, color has many attributes. Those include characteristics such as harmony — when two or more colors are brought together and produce a satisfying effective response and temperature.

So, we can say that color is a sensation, a human reaction to a hue arising in part from the optic nerve, in part from education and exposure to color, and perhaps in the largest part, simply from the human senses.

[source of definition](#)

EDUCATIONAL AIMS	TARGET PUBLIC	NECESSARY ABILITIES	MATHEMATICAL AND ARTISTIC COMPETENCIES
<ul style="list-style-type: none"><li>- to learn about color-math problems</li><li>- to learn about the right amount (proportion) of certain colors in a picture</li></ul>	<ul style="list-style-type: none"><li>- students aged 15 to 19</li><li>- high school</li></ul>	<ul style="list-style-type: none"><li>- vision</li><li>- love for Art</li><li>- coloring skills</li></ul>	<ul style="list-style-type: none"><li>- Discover and learn to apply the four color theorem</li><li>- Learn about artists Jerzy Grabowski and Piet Mondrian and their work</li></ul>

## LEARNING ACTIVITIES

### Learning Activity 1

#### Four color theorem

- students will discover and learn about Four color theorem

Materials: examples for coloring, colors, paper, geometric accessories

Activity details:

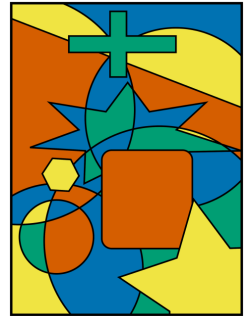
- teachers prepare for students worksheets with examples for coloring and a question: how many minimum of colors do they need to paint the pattern so that adjacent sections are not painted the same color
- students discover that every sample (regardless of size) can be colored with minimum of four colors
- students make samples of their own and color them

Links:

<https://goranagnjdicmath.files.wordpress.com/2015/06/problem-bojanja.pdf>

[https://en.wikipedia.org/wiki/Four\\_color\\_theorem](https://en.wikipedia.org/wiki/Four_color_theorem)

[https://bit.ly/coloring\\_problem](https://bit.ly/coloring_problem)



### Learning Activity 2

#### Jerzy Grabowski Art

- students will learn about Jerzy Grabowski's geometrical abstract art
- students will try to recreate similar artworks

Materials: paper, colors, geometric accessories

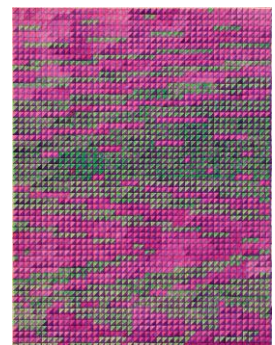
Activity details:

- students learn about Jerzy Grabowski's geometrical abstract art
- students recreate his artworks using wooden colors, geometric accessories and notebook papers

Links:

<https://www.retroavangarda.com/math-and-color-geometrical-abstract-art-of-jerzy-grabowski/>

<https://desa.pl/pl/artysci/jerzy-grabowski/>



### **Learning Activity 3**

#### **Piet Mondrian Art**

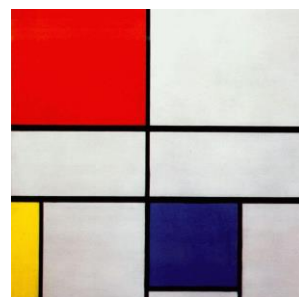
- students will learn about Piet Mondrian's abstract art
- students will try to recreate similar artworks

Materials: paper, colors, geometric accessories

Activity details:

- students learn about Piet Mondrian's abstract art
- students recreate his artworks using different types of colors, geometric accessories, and white papers

Links: <https://www.piet-mondrian.org/>



### **Learning Activity 4**

#### **Fibonacci spiral**

- students will learn about Fibonacci sequence, thereby about Fibonacci number, Fibonacci spiral and Golden Ratio
- students will create Fibonacci spirals and color them:
  - with four colors
  - in Jerzy Grabowski style
  - in Piet Mondrian style
  - as they like

Materials: paper, colors, geometric accessories

Activity details:

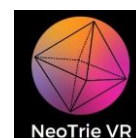
- students learn about Fibonacci sequence, thereby about Fibonacci number, Fibonacci spiral and Golden Ratio
- students draw Fibonacci spirals and color them in various ways

Links: <https://www.youtube.com/watch?v=6QBeKyznwk0>  
<https://www.geogebra.org/m/t9g3534c>



#### **Extra activity in Neotrie VR**

Free game with colors in multiplayer mode.  
<https://www2.ual.es/neotrie/project/geomview/#Coloring>



## Bibliography:

[https://bit.ly/coloring\\_problem](https://bit.ly/coloring_problem)

<https://goranagnjidicmath.files.wordpress.com/2015/06/problem-bojanja.pdf>

[https://en.wikipedia.org/wiki/Four\\_color\\_theorem](https://en.wikipedia.org/wiki/Four_color_theorem)

<https://www.retroavangarda.com/math-and-color-geometrical-abstract-art-of-jerzy-grabowski>

<https://desa.pl/pl/artysci/jerzy-grabowski>

<https://www.piet-mondrian.org>

<https://www.youtube.com/watch?v=6QBeKyznwk0>

<https://www.geogebra.org/m/t9g3534c>

## Results

[https://bit.ly/coloring\\_problem](https://bit.ly/coloring_problem)

[https://bit.ly/four\\_color\\_theorem](https://bit.ly/four_color_theorem)

[https://bit.ly/mondrian\\_students\\_work](https://bit.ly/mondrian_students_work)

[https://bit.ly/grabowski\\_students\\_work](https://bit.ly/grabowski_students_work)

[https://bit.ly/Fibonacci\\_spiral\\_students\\_work](https://bit.ly/Fibonacci_spiral_students_work)





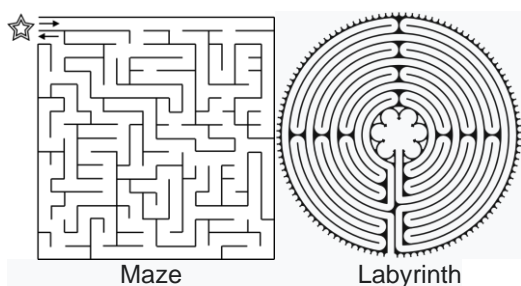
Co-funded by the  
Erasmus+ Programme  
of the European Union

2020-1-FR01-KA227-SCH-095534



## LABYRINTH

Maze and labyrinth depict a complex and confusing series of pathways.



[images source](#)

A maze is a confusing pathway that has many branches, choices of path and dead-ends.

A labyrinth has a single through-route with twists and turns but without branches. There is just one path from the entrance to the center.

[source of definition](#)

EDUCATIONAL AIMS	TARGET PUBLIC	NECESSARY ABILITIES	MATHEMATICAL AND ARTISTIC COMPETENCIES
<ul style="list-style-type: none"><li>- to learn about difference between labyrinth and maze</li><li>- to construct own maze and concentric labyrinth</li></ul>	<ul style="list-style-type: none"><li>- students aged 15 to 19</li><li>- high school</li></ul>	<ul style="list-style-type: none"><li>- vision</li><li>- love for Art</li><li>- stitch skills</li></ul>	<ul style="list-style-type: none"><li>- spot geometric shapes on real mazes and labyrinths</li><li>- draw a labyrinth by constructing concentric circles</li><li>- draw a maze on a square grid and make a model of a maze garden</li></ul>

# LEARNING ACTIVITIES

## **Learning Activity 1**

- The teacher explains and presents to the students the difference between a maze and a labyrinth

Materials: Internet, PC,...

Activity details:

- the teacher prepares the presentation
- the teacher explains to the students the difference between a maze and a labyrinth
- the teacher shows students photos of real labyrinths and mazes (in cathedrals, gardens,...)
- In examples of labyrinths, students notice various geometric shapes

Links:

[https://www.diffen.com/difference/Labyrinth\\_vs\\_Maze](https://www.diffen.com/difference/Labyrinth_vs_Maze)

<https://en.wikipedia.org/wiki/Labyrinth>

[https://en.wikipedia.org/wiki/Hedge\\_maze](https://en.wikipedia.org/wiki/Hedge_maze)

## **Learning Activity 2**

Finger concentric labyrinth

- Students draw a concentric labyrinth
- Students decorate the drawn labyrinth

Materials: paper, pencil, ruler, eraser, scissors, compass, protractor, colored pencils, acrylic paint,...

Activity details:

- Students draw a concentric labyrinth on paper modeled after the labyrinth in Chartres Cathedral
- According to the drawn labyrinth, the students decorate the drawn labyrinth

Links:

<https://www.youtube.com/watch?v=nKEvP-vA4P0>

### **Learning Activity 3**

- Students draw a maze and make a model of the maze garden

Materials: paper, pencil, ruler, glue ...

Activity details:

- The teacher teaches the students about Kruskal's and Prim's algorithm
- using the mentioned algorithms, the students draw a plan of the maze
- Students make paper cubes and stick them on the drawn maze to get a model of the maze garden

Links:

<https://repozitorij.pmf.unizg.hr/islandora/object/pmf%3A5515/datastream/PDF/view>  
<https://www.pinterest.com/search/pins/?q=garden%20maze%20model&rs=rs&eq=&etslf=1862>

### **Extra activity in Neotrie VR**

Students design a labyrinth and play in multiplayer mode.

<https://www2.ual.es/neotrie/project/geomview/#Labyrinths>



### **Bibliography:**

[https://www.diffen.com/difference/Labyrinth\\_vs\\_Maze](https://www.diffen.com/difference/Labyrinth_vs_Maze)  
<https://en.wikipedia.org/wiki/Labyrinth>  
[https://en.wikipedia.org/wiki/Hedge\\_maze](https://en.wikipedia.org/wiki/Hedge_maze)  
<https://www.youtube.com/watch?v=nKEvP-vA4P0>  
<https://repozitorij.pmf.unizg.hr/islandora/object/pmf%3A5515/datastream/PDF/view>  
<https://www.pinterest.com/search/pins/?q=garden%20maze%20model&rs=rs&eq=&etslf=1862>

### **Results**

[https://bit.ly/labyrinth\\_students](https://bit.ly/labyrinth_students)

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## GEOMETRICIAN'S VIEWS



### Illusions



One of the oldest examples of optical illusion in art, this ancient sculpture shows a bull and an elephant sharing a common head. Made over 850 years ago, this bas-relief can be found at the Airavatesvara Temple in India.

Geometrical-optical illusions are visual illusions, also optical illusions, in which the geometrical properties of what is seen differ from those of the corresponding objects in the visual field.

### Explanation

Optical illusions are pictures that play tricks on our eyes and baffle our perception. They are not the result of faulty vision. Depending on light, viewing angle, or the way the picture is drawn, we may see things that aren't there – and

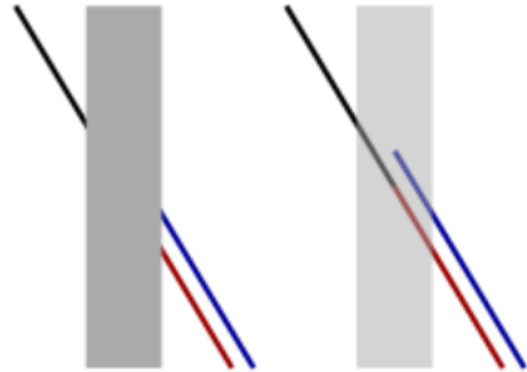
often don't see what's right under our nose. These tricks of the eye and mind have been part of human experience since the beginning of history. The ancient Greeks made use of optical illusions to perfect the appearance of their great temples. In the Middle Ages, misplaced perspective was occasionally incorporated into paintings for practical reasons. In more recent times, many more illusions have been created and implemented in the graphic arts.

# Examples of geometrical-optical *illusions*

A few examples are drawn from the list of optical illusions.

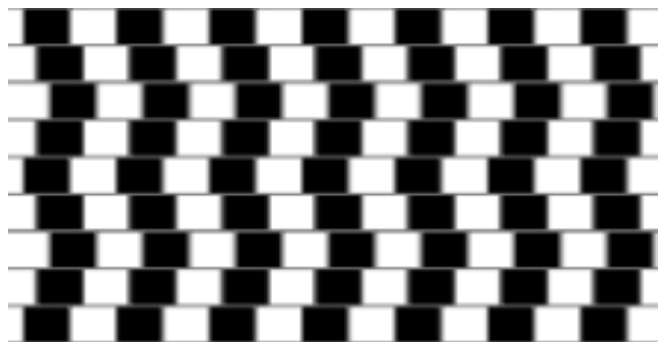
They illustrate illusions of position  
([Poggendorff illusion](#))

A straight black and red line is obscured by a gray rectangle. The blue line, rather than the red line, appears to be a continuation of the black one, which is clearly shown not to be the case on the second picture. Instead there is an apparent position shift of the lower portion of the line



They illustrate illusions of orientation

The **café wall illusion** is a [geometrical-optical illusion](#) in which the [parallel](#) straight dividing lines between staggered rows with alternating black and white "bricks" appear to be sloped, not parallel as they really are.



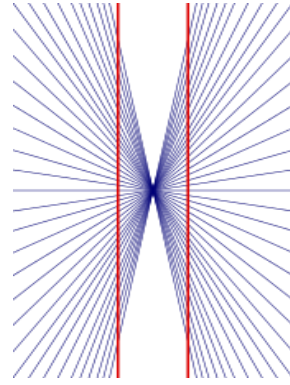
Optical Illusion Building, Melbourne





They illustrate illusions of rectilinearity or straightness of lines

When two straight and parallel lines are presented in front of radial background (like the spokes of a bicycle), the lines appear as if they were bowed outwards



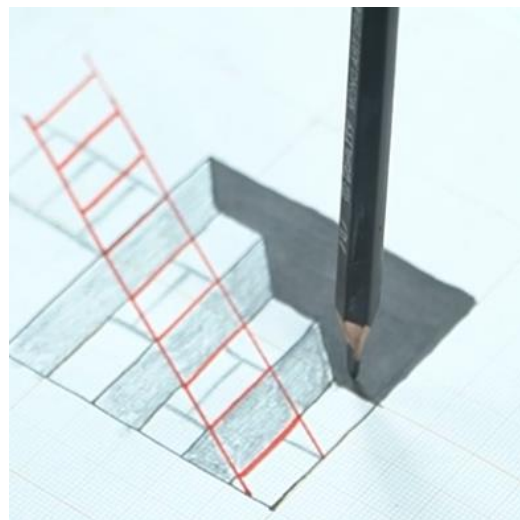
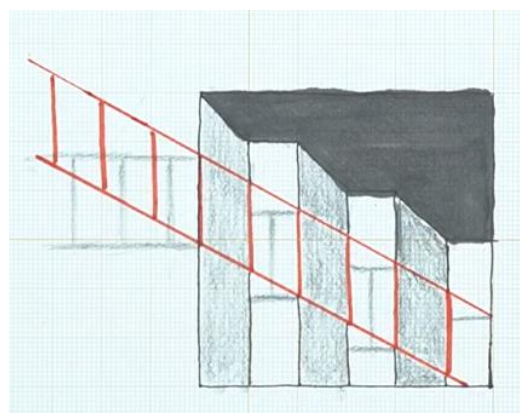
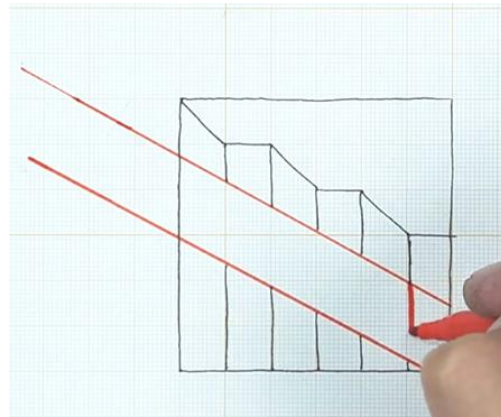
This construction has only 4 sticks and appeared 4 triangles

### Activity 1:

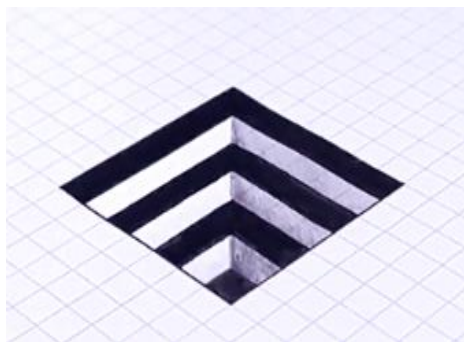
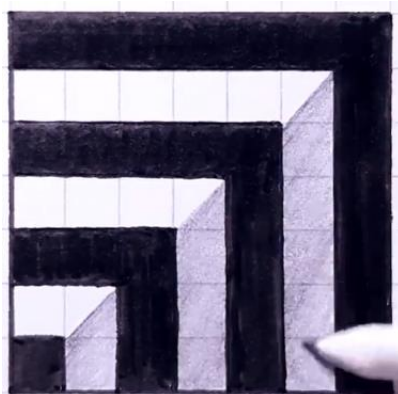
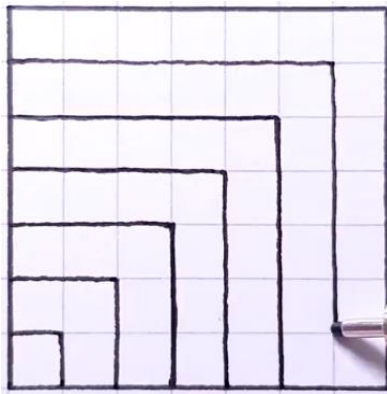
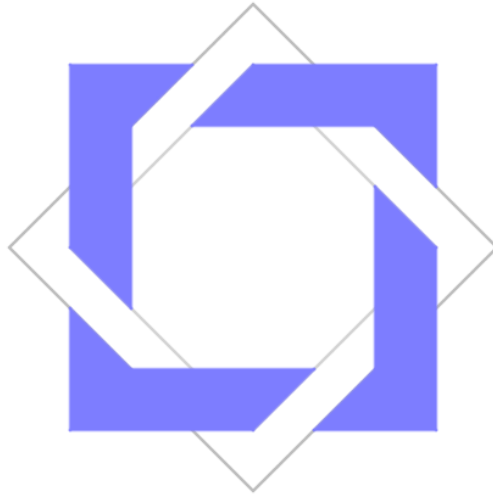
Take 4 sticks with the same material, find a property place with a good shadow

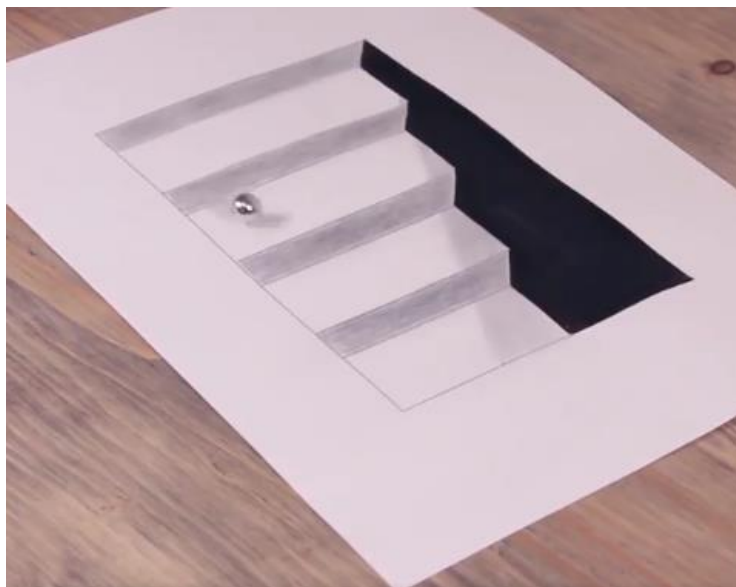
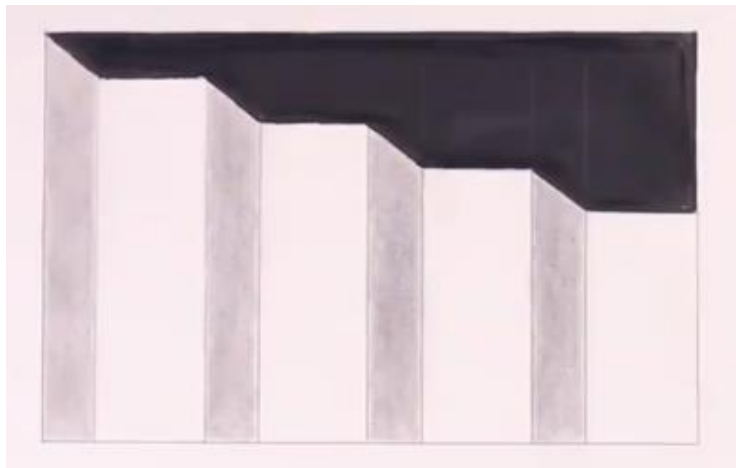
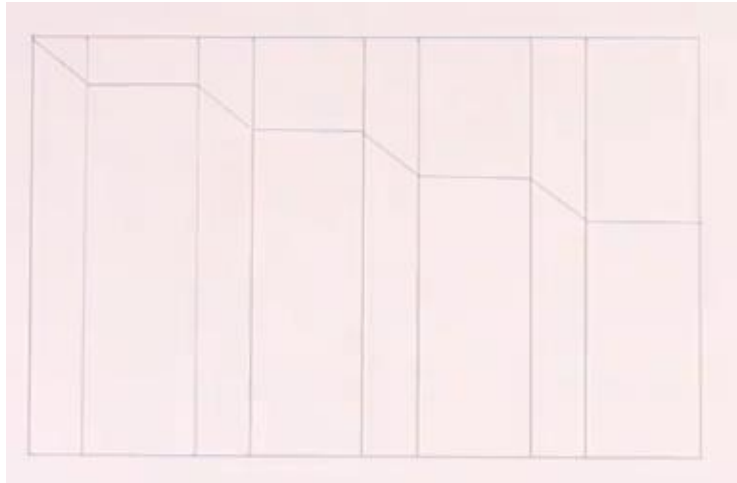


**Activity 2:** paint on paper or draw in geogebra the follows

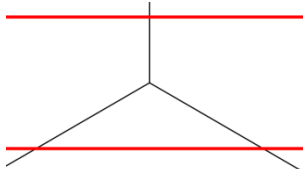
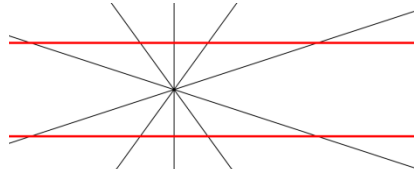
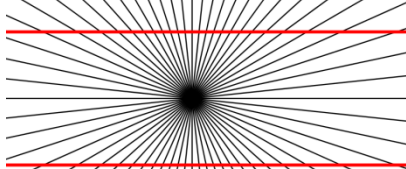


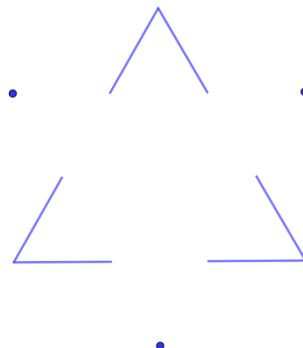
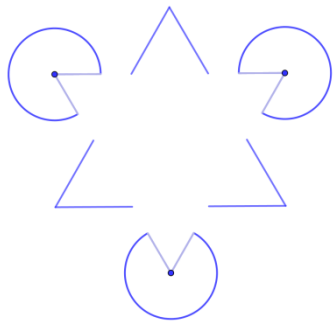
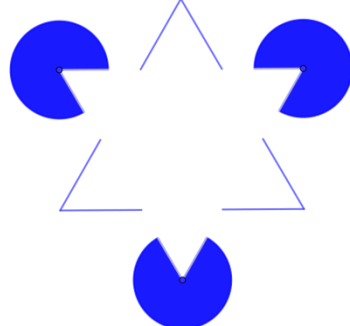
*Geogebra's construction*

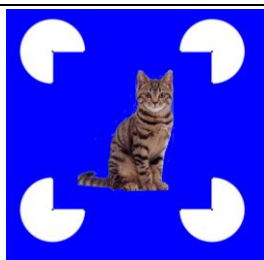
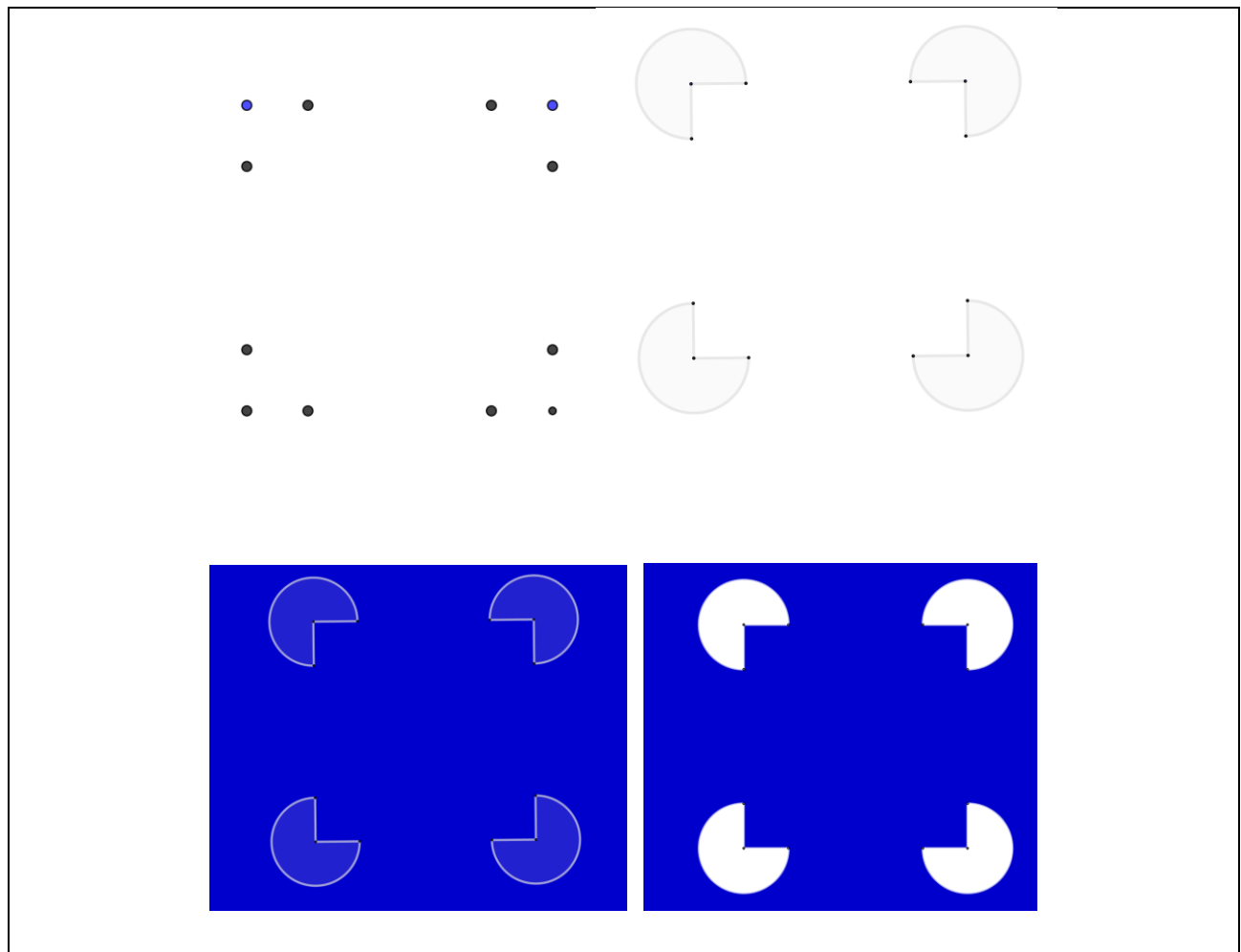




**The Hering illusion** (1861): When two straight and parallel lines are presented in front of radial background (like the spokes of a bicycle), the lines appear as if they were not parallel (bowed outwards)

		
Sequence of $i=3$ Rays, $n=3$	Sequence of $i=10$ Rays, $n=10$	Sequence of $i=16$ Rays, $n=60$
<p><b>Activity 3:</b>          Use the geogebra software to construct the Hering illusion  <u>Instructions</u> for the construction          Make two sliders <math>k=\text{slider1}</math>, , <math>n=\text{slider3}</math> (1-30)          In the input box type:  <math>y=k</math>, <math>y=-k</math>, (for two parallel lines), make them with a red color          Take points <math>A(0,0)</math>, <math>B(2,0)</math> for Ray's construction,          In the input box type:          Sequence(Ray(A, Rotate(B, <math>i * 360^\circ / n</math>, A)), <math>i</math>, 1, <math>n</math>)</p>		

<p style="text-align: center;"><b>Kanizsa triangle</b></p> <p>Created by Italian psychologist Gaetano Kanizsa in 1955, the Kanizsa Triangle is a classic example of illusory contours. As shown in the image above, a white equilateral triangle can be clearly perceived even though there are no explicit lines or enclosed spaces to indicate such a triangle. The white triangle actually appears brighter than the surrounding white background, even in areas where there is no change in luminance or color</p>		
		



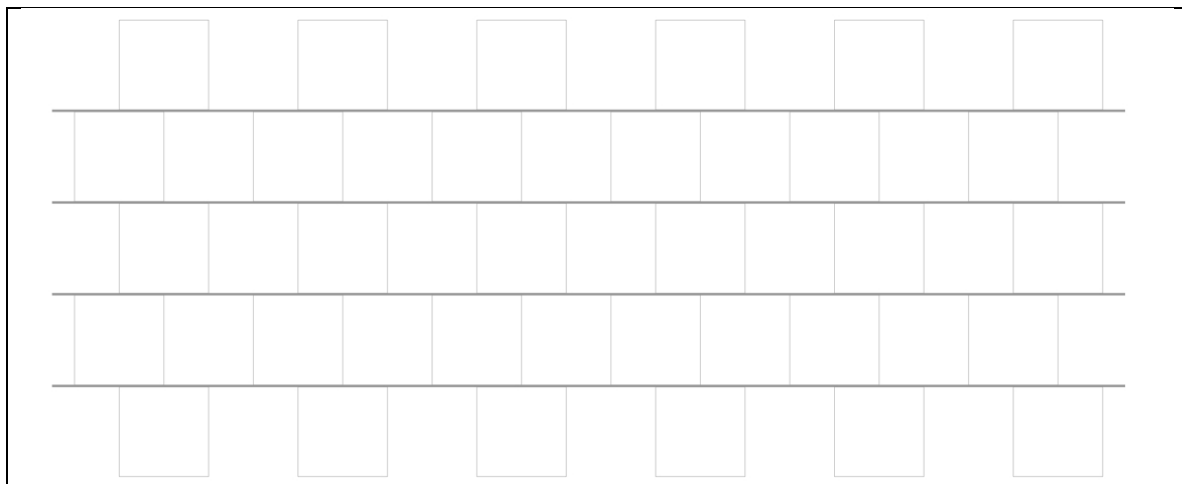
A new study published in Applied Animal Behavior Science found cats were more likely to sit inside 2-D shapes that imitate an illusion of a square, and it may give researchers more insight into our furry friend's perception of visual illusions (The study is cheekily titled "If I fits I sits: A citizen science investigation into illusory contour susceptibility in domestic cats.") [Here is the article](#)

#### **Activity 4:** Do in Geogebra

Draw a square, keep only the 4 vertices, construct a circle (Center each vertex, radius=1)

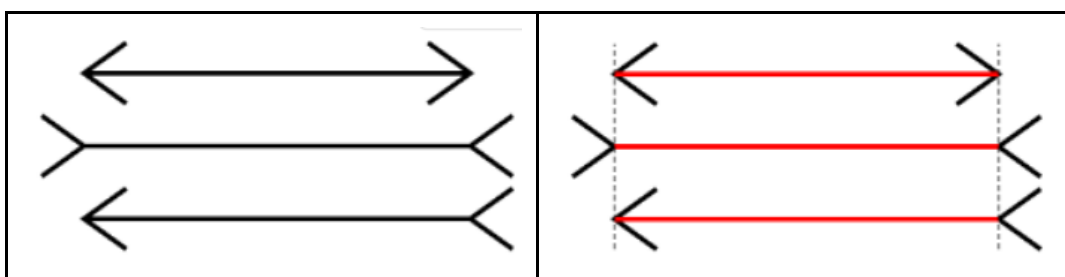
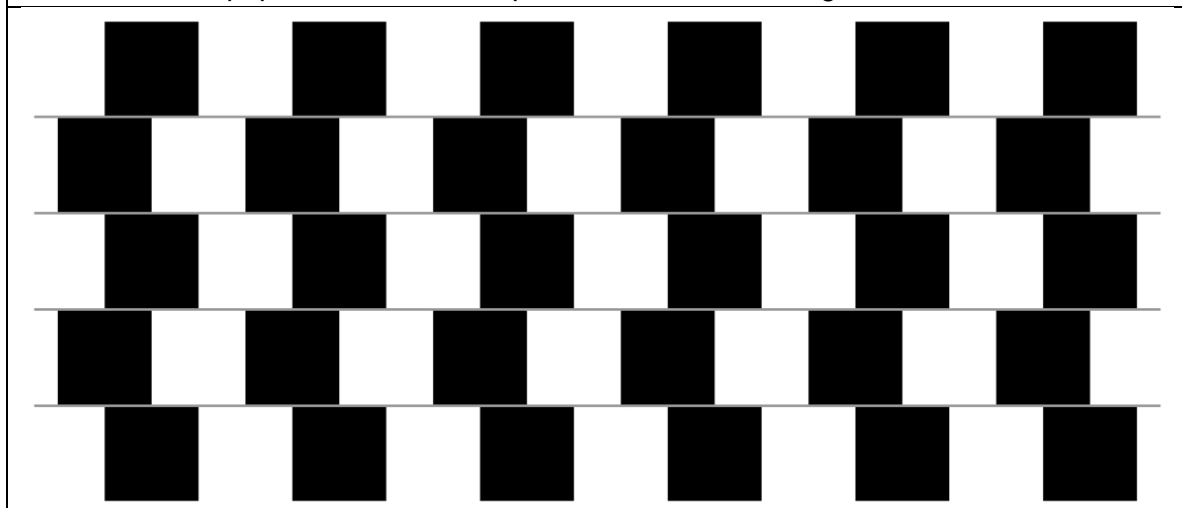
Draw 4 circular sectors, change the colors of objects, and the background colors as in the above pictures. Find a cat's figure, try to insert it into illusion's square



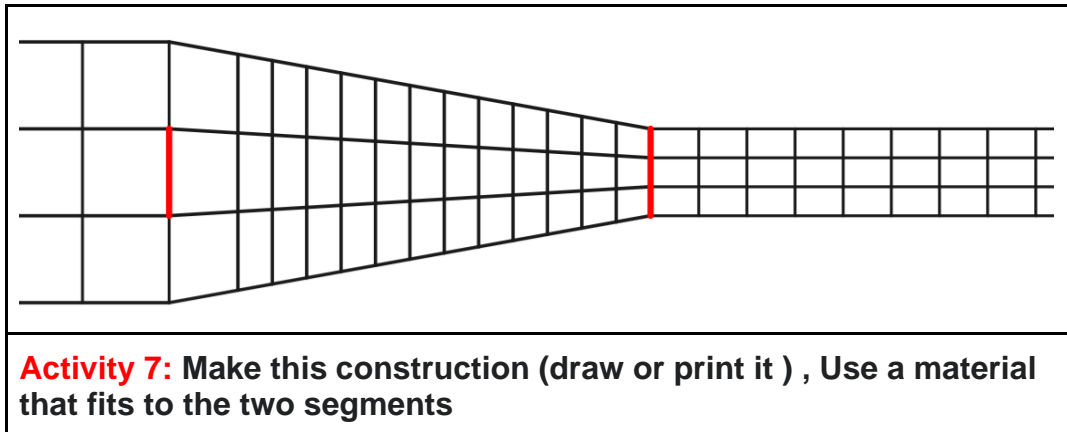


**Activity 5 :**

Print the above paper and draw the squares as the below image



**Activity 6:** Make this construction (draw or print it) , Use a material that fits to all segments, or use two parallel materials as in the above figure



## Webography

<https://www.nationalgeographic.com/science/article/partner-content-when-circles-are-finally-squared>

<https://www.youtube.com/watch?v=0RsrQDGUIIq>

<https://plus.maths.org/content/visual-curiosities-and-mathematical-paradoxes>

<http://www.whatispsychology.biz/kanizsa-triangle-illusion-explanation>

<https://medium.com/archilyse/1-how-architecture-photography-can-trick-our-perception-e3391ae1063>

<https://www.facebook.com/watch/?v=1044105369658579>

<https://www.facebook.com/ArtRevolutionTV/>

<http://www.psy.ritsumei.ac.jp/~akitaoka/nuriee.html>

[https://www.smithsonianmag.com/smart-news/if-i-fits-i-sits-experiment-shows-felines-also-sit-illusory-boxes-](https://www.smithsonianmag.com/smart-news/if-i-fits-i-sits-experiment-shows-felines-also-sit-illusory-boxes-180977681/?fbclid=IwAR2NsX93oyENJjKUQDDs5xOIhDIInzBR4s60N-zCudEZzc89ZTFhP7dYfXfg)

[180977681/?fbclid=IwAR2NsX93oyENJjKUQDDs5xOIhDIInzBR4s60N-zCudEZzc89ZTFhP7dYfXfg](https://www.smithsonianmag.com/smart-news/if-i-fits-i-sits-experiment-shows-felines-also-sit-illusory-boxes-180977681/?fbclid=IwAR2NsX93oyENJjKUQDDs5xOIhDIInzBR4s60N-zCudEZzc89ZTFhP7dYfXfg)

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GEOMETRICIAN'S  
VIEWS



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## BRAIDS

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A **braid** (also referred to as a **plait**) is a complex structure or pattern formed by interlacing two or more strands of flexible material such as textile yarns, wire, or hair

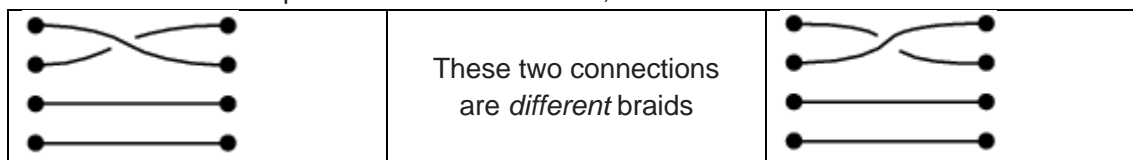
Braids are everywhere, in hairdressing, in jewelry, in leather belts, in ropes, in cakes, in bread and many other objects. Braiding is one of the oldest ways to decorate objects





A creation of a basic braid using three strings

In mathematics, the braid group on  $n$  strands  $B_n$  also known as the Artin braid group, is the group whose elements are equivalence classes of  $n$ -braids, and whose group operation is composition of braids


Consider two sets of four items lying on a table, with the items in each set being arranged in a vertical line, and such that one set sits next to the other. (In the illustrations below, these are the black dots.) Using four strands, each item of the first set is connected with an item of the second set so that a one-to-one correspondence results. Such a connection is called a *braid*. Often some strands will have to pass over or under others, and this is crucial



But two such connections which can be made to look the same by "pulling the strands" are considered *the same* braid:

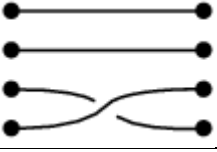
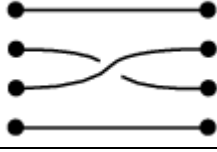
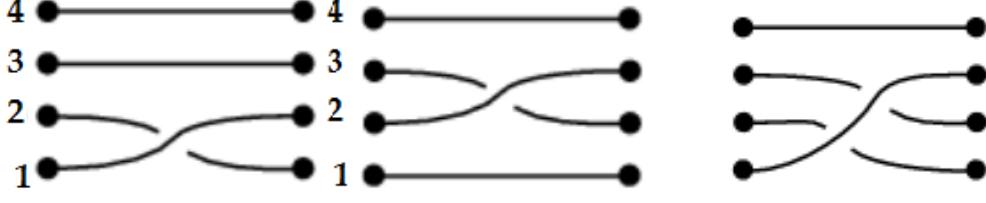
	The left braid is the same as the right	
---	---	--

All strands are required to move from left to right; knots like the following are *not* considered braids:



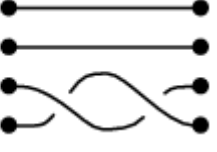
	It is not a braid
---	-------------------

## Compositions of braids

Any two braids can be *composed* by drawing the first next to the second, identifying the four items in the middle, and connecting corresponding strands:

	composed with	
<p>Compositions of two braids yields another braid</p> 		

The Compositions of braids is an operation like the product of positive numbers  $a, b$

		=	
$a \cdot b$	$a, b \in \mathbb{R}^+$		

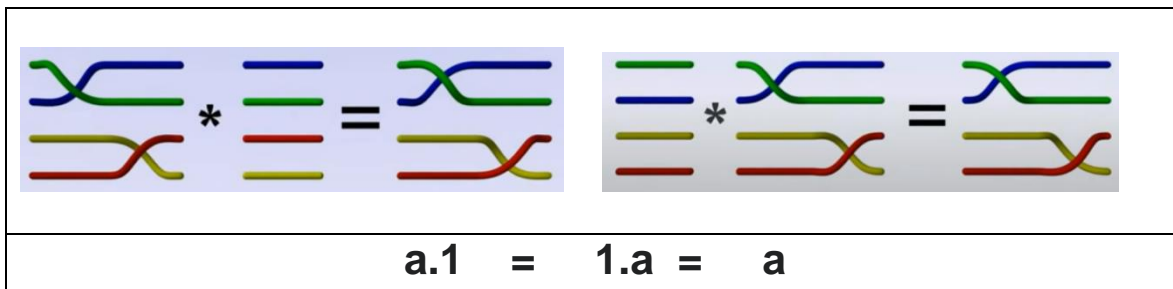
## Group structure

The above composition of braids is a group operation.

The associativity property holds  $(ab)c=a(bc)$

The commutativity does not holds  $ab \neq ba$

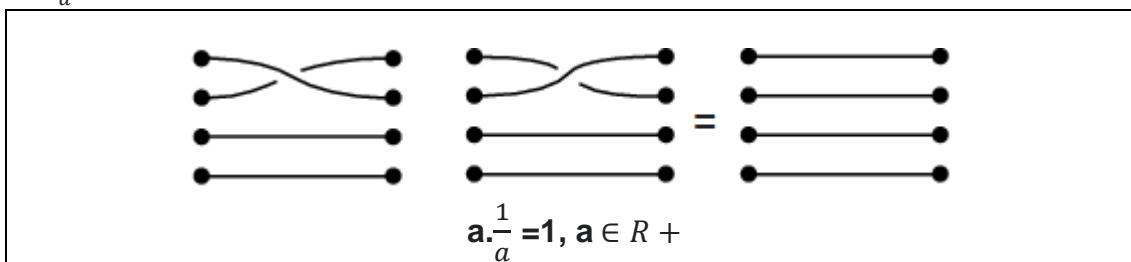
The identity element is the braid consisting of four parallel horizontal strands.



The inverse of a braid consists of that braid which "undoes" whatever the first braid did, which is obtained by flipping a diagram across a vertical line going through its centre (like a mirror)

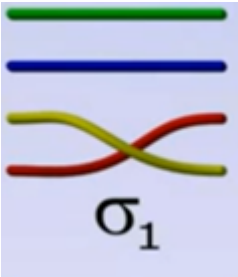
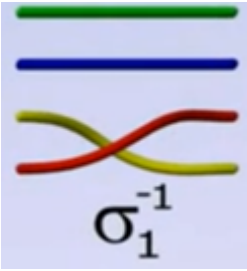
Any braid (number) has an inverse with respect to the composition (product)

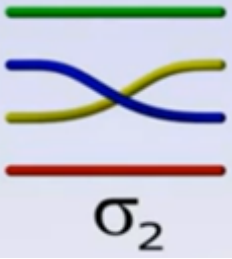
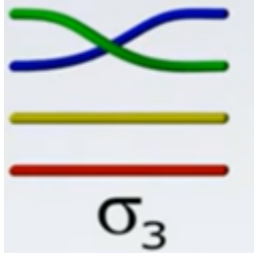
$$a \cdot \frac{1}{a} = 1, a \in R +$$



### *Emil Artin and his theory*

The first Mathematician who studied braids in this terms was Emil Artin. In 1928 he wrote the first paper about braids in German. Twenty years later he wrote a second important one, in English. He was the first who described the structure of the group on the set of braids and noted that group can be described in the follow way: Any braid can be described as a word in the  $\sigma_i$ 's and their inverses. In every group we have these abductions. They are local: they change just a little part of the word, keeping the rest fixed

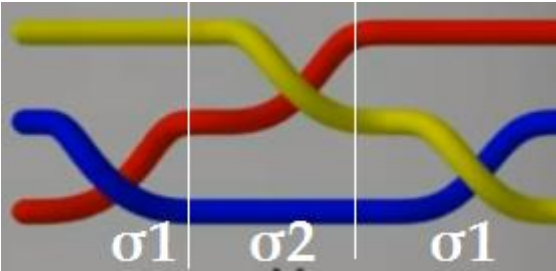
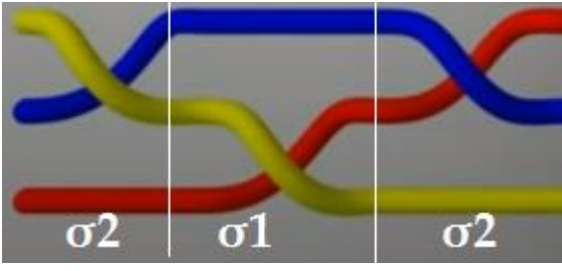
 <p style="text-align: center;">"1&lt;2")</p> <p>The first strand (yellow) is back of the first (red)</p>	 <p style="text-align: center;">"1&gt;2"</p> <p>The second strand (yellow) is back of the first (red)</p>

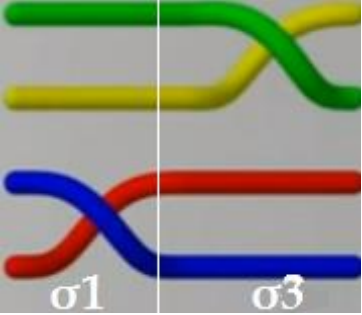
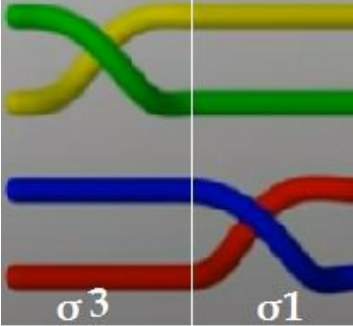
 <p>The second strand (yellow) is back of the third (blue)</p>	 <p>The third strand (blue) is back of the fourth (green)</p>
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In this theory there are two important properties ( or relations or moves) that allow substitutions. So we can generate all the words that describe the same braid. We have turned braids into mathematical objects.

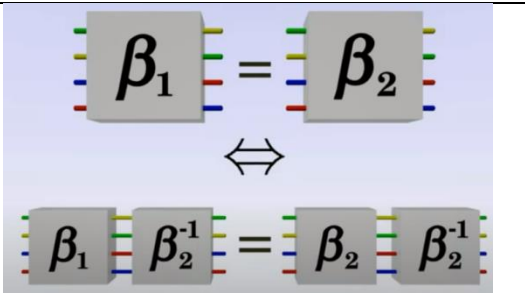
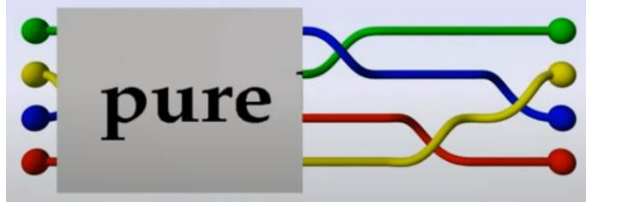
**Theorem:** The braid group on  $n$  strands admits a presentation with generators  $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$  and relations:

- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$  , for  $1 \leq i \leq n-2$
- $\sigma_i \sigma_j = \sigma_j \sigma_i$  , for  $|i-j| > 1$

	=	
Moving the blue strand through a crossing and exchanging the left and right crossing then the word $\sigma_2 \sigma_1 \sigma_2$ equals to the word $\sigma_1 \sigma_2 \sigma_1$		

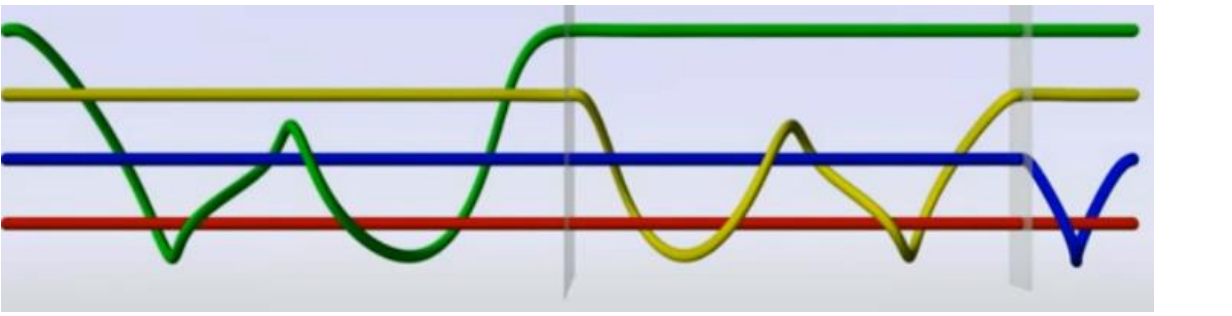
	=	
Moving the blue strand through a crossing and exchanging the left and right crossing then the word $\sigma_1 \sigma_2$ equals to the word $\sigma_2 \sigma_1$		

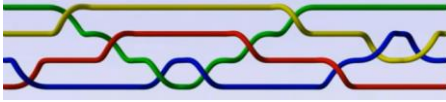
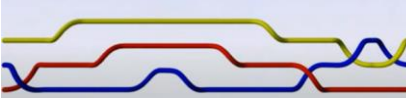
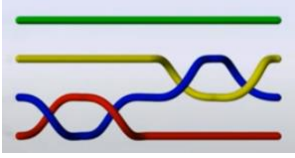


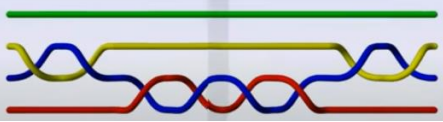

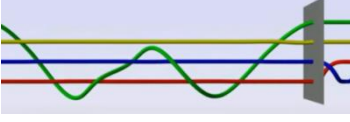
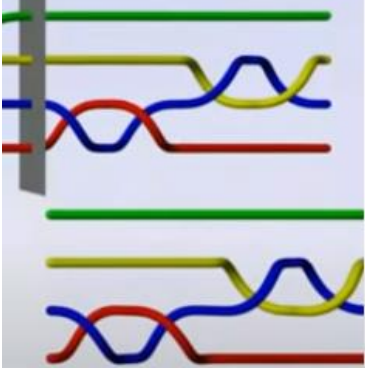
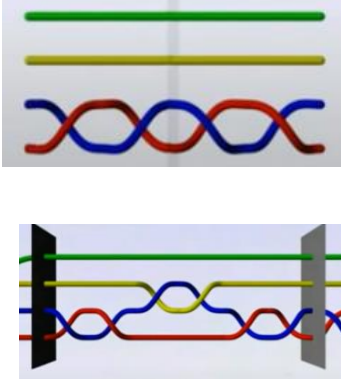
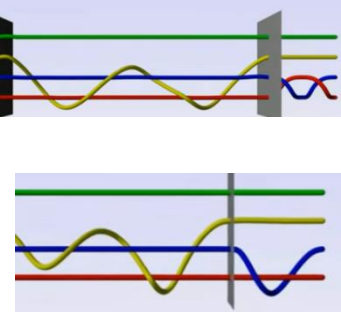
	<p>If the bubbles at the same level , on the left and on the right have the same color the the braid called pure</p> 
<p>Two words represent the same braid if and only if one, composed with the inverse of the other, is equivalent to the trivial braid</p>	

We consider only pure braids in this chapter

### Decomposed a pure braid to a normal form

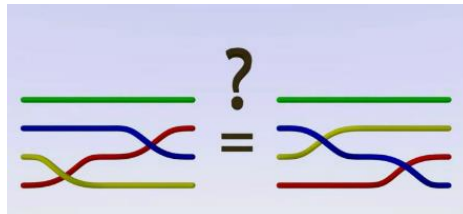
		
<p>In the 1<sup>st</sup> block the 4<sup>th</sup> strand (green) can link to the others straight strands (1<sup>st</sup> to 3<sup>rd</sup> )</p>	<p>In the 2<sup>st</sup> block the 3<sup>th</sup> strand (yellow) link to the others straight strands (1<sup>st</sup> , 2<sup>rd</sup> ,4<sup>th</sup>)</p>	<p>And so on, until the last block, in which the 1<sup>st</sup> strand can link to the other straight strands</p>

		
<p>1. Here is an non trivial braid</p>	<p>2. First we delete the 4<sup>th</sup> strand (green)</p>	<p>3. Then we replace this one with a trivial one</p>

		
<p>4. We compose this braid with its inverse, getting the identity braid. So composing the original braid with it, we don't change the braid</p>	<p>5. On the left we have a braid that can be put into the desired form, with all strands straight except the 4<sup>th</sup> (green)</p>	<p>6. Here is the desired 1<sup>st</sup> block of a normal braid</p>
		
<p>7. We do the same procedure on the right side</p>	<p>8. We delete the 3<sup>rd</sup> (yellow) strand and replace it with a trivial one steps 2, 3 and 4. Here is the 2<sup>nd</sup> block</p>	<p>9a) The second block is combed 9b) The braid is now in a normal form</p>
<p>Artin has proved that a braid is trivial if and only if, once it is combed, every block is trivial, meaning that it can be represented with no crossings at all. So the above braid is not trivial</p>		

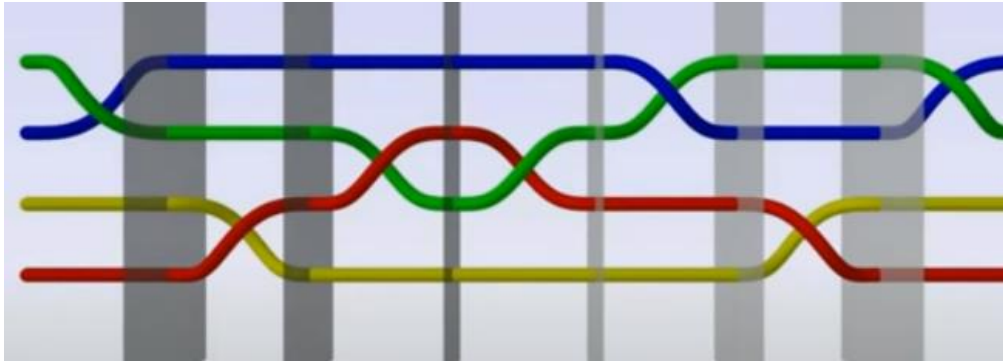
## Activities

**Activity1:** Are the two braids equal?



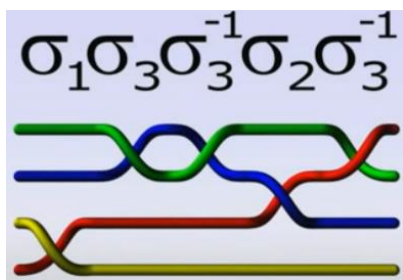
**Activity2:** Can you find the word that describes the follow braid?

that describes

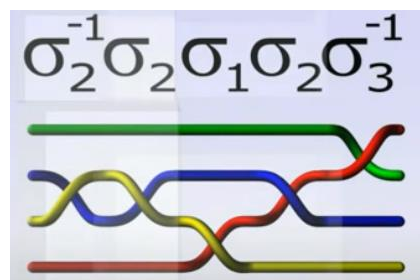


**Activity3:** Using the follow two important properties prove the equality in the follow figure:

- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ , for  $1 \leq i \leq n-2$
- $\sigma_i \sigma_j = \sigma_j \sigma_i$ , for  $|i-j| > 1$



=



**Activity4:**

$$\begin{aligned} \sigma_1 \sigma^{-1}_2 &= 1 \quad \sigma_1 \sigma^{-1}_2 1 = \sigma_1 \sigma^{-1}_1 \sigma_1 \sigma^{-1}_2 \sigma_2 \sigma^{-1}_2 = \sigma_1 \sigma_1 \sigma_2 \sigma^{-1}_1 \sigma^{-1}_2 \sigma^{-1}_2 = \sigma_1 \sigma_2 \sigma_1 \sigma^{-1}_2 \sigma^{-1}_1 \sigma^{-1}_2 \\ &= \sigma_2 \sigma_1 \sigma_2 \sigma^{-1}_2 \sigma^{-1}_1 \sigma^{-1}_2 = \sigma_2 \sigma^{-1}_2 \sigma_2 \sigma^{-1}_2 \sigma_1 \sigma^{-1}_1 = 1.1.1 = 1 \end{aligned}$$

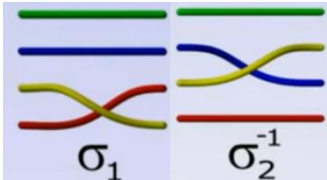
Fill the follow table

print the cards or copy the images and paste them until you prove that  $\sigma_1 \sigma^{-1}_2 = 1$

Take the sheet and the cards:

[https://docs.google.com/document/d/1-f3Xx73TenRaPHFTGi\\_-](https://docs.google.com/document/d/1-f3Xx73TenRaPHFTGi_-GpYXWJwJv13N/edit?usp=sharing&oid=110908192210660401531&rtpof=true&sd=true)

[GpYXWJwJv13N/edit?usp=sharing&oid=110908192210660401531&rtpof=true&sd=true](https://docs.google.com/document/d/1-f3Xx73TenRaPHFTGi_-GpYXWJwJv13N/edit?usp=sharing&oid=110908192210660401531&rtpof=true&sd=true)


$\sigma_1 \sigma_2^{-1} =$
$1 \sigma_1 \sigma_2^{-1} 1 =$
$\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1} \sigma_2 \sigma_1^{-1} =$
$\sigma_1 \sigma_2 \sigma_1 \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1} =$
$\sigma_2 \sigma_1 \sigma_2 \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1} =$
$\sigma_2 \sigma_2^{-1} \sigma_2 \sigma_2^{-1} \sigma_1 \sigma_1^{-1} =$
$1.1.1 = 1$

## Webography

<http://matematita.science.unitn.it/braids/faq.html>

<https://www.imaginary.org/film/braids>

<https://www.youtube.com/watch?v=VhryJaoJT1Q>

[http://eprints-phd.biblio.unitn.it/663/1/dalvit\\_popularization\\_braid.pdf](http://eprints-phd.biblio.unitn.it/663/1/dalvit_popularization_braid.pdf)



# Paintings: Mondrian

Mathematics, Nature, Science, Art = Beauty

Mondrian is a very famous painter for his extremely abstract art. He is inspiration to many other artists. The one the most known piece of art is his painting “Composition”. It was painted in 1930 and greatly contributed to visual language despite using small canvases. It combines geometrical figures, which can be an excellent connection to geometrical lessons. Mondrian was also an inspiration to many other artists in the fashion and design industry.

During this lesson, students will learn the basics of geometry, the connection between mathematics and art, and they also learn about Mondrian and his art direction. The lesson also enhances students’ creativity. Students can explore mathematical concept of calculating areas of squares and rectangles, but also explore combination of colors and shades.

## 1 EDUCATIONAL AIMS

Students will learn basics of geometry and learning about Mondrian and his art direction.

## 2 TARGET PUBLIC

Pupils aged **14-16**  
Upper secondary school

## 3 SKILLS NEEDED

Basic knowledge of geometry.

## 4 MATHEMATICAL AND ARTISTIC AIM

The art of Mondrian was used to show students that there is strong connection between mathematics and art.

## Learning activities

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### Learning activity 1:

Pupil age: 14-16 years old

Material: pencil, formula chart

Activity's detail: Students learned about mathematical properties of geometrical objects like triangles and quadrilaterals. Students revise calculating area of square and rectangle using formulas. Students will understand how to calculate area as the size of space of 2-dimenzional objects.

Students work: Students solve different tasks, such as:

Find the area of the given A) square of side 5.5 cm, B) square of side 0.01 cm, C) rectangle of lenght 8cm and breadht 3cm D) rectangle of lenght 9mm and breadht 30cm.

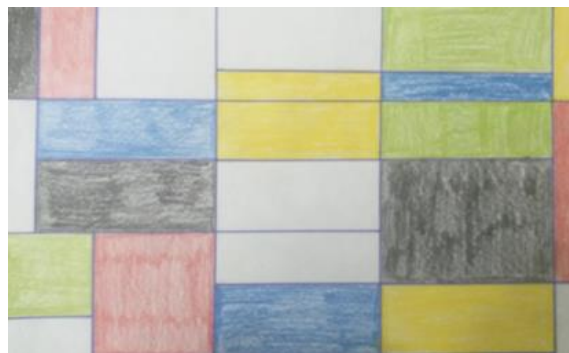
### Learning activity 2:

Pupil age: 14-16 years old

Material: Color pencils, rulers

Activity's detail: Based on what they learned students create their own work inspired by mathematics and Mondrian's art. Students research about art of Mondrian. Students' task was to make a Mondrian-like inspire artwork.

Students' work:





## References

1. <https://www.britannica.com/biography/Piet-Mondrian/Later-years>
2. <https://www.tate.org.uk/kids/explore/who-is/who-piet-mondrian>



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# CURVES

## 1. Introduction

In mathematics, a curve, or curved line, is an object of the usual plane or space, similar to a straight line but not necessarily linear. For example, circles, lines, segments, and polygonal lines are curves.

In art, the curve can imitate Nature or marry the gesture of the hand which spontaneously does not completely master the straight line, to follow a natural momentum.

Art Nouveau is an artistic movement of the late 19th and early 20th centuries that relies on the aesthetics of curved lines.

## 2. Summary of the activity

- Educational goal: Discover curves
- Target Age: from 5 years old
- Skill required: know how to find the middle of a segment
- Mathematical and artistic goal, build curves, invent and create

## LEARNING ACTIVITIES

### Activity 1: Straight and Curved

Age of students: from 5 years old

Working mode: individual

Objective: Distinguish straight and curved

Course of the activity:

1. Understand the difference between straight lines and curves
2. Make a man made with curves

Student work:

### Activity 2: Discovery of Art Nouveau

Age of students: from 9/10 years old

Objective ; Discover Art Nouveau

Work modality: class group

Course of the activity: Visit of the Ecole de Nancy

### **Activity 3: Using Scratch and curve**

Age of students: From 14 years old

Working mode: individual

Equipment: computer with Scratch

Objective: draw a curve representative of a function with Scratch

Course of the activity:

Proposal to plot the representative curve of the function  $f$  defined by  $f(x)=x^2$

Programming with Scratch

Plot of the curve representative of the function  $f$  defined by  $f(x)=x^2$

Student work:

### **Activity 4: Using dynamic geometry software and curves**

Age of students: From 16 years old

Working mode: individual

Equipment: computer with Geogebra software

Objective: draw curves representative of functions with dynamic geometry software

Course of the activity:

Draw 4 curves

Student work:

Using dynamic geometry software, the students will draw the 4 curves to form the word LOVE

### **Activity 5: Bézier curves**

Age of students: from 15 years old

Working mode: in group

Material: Computer if slideshow and A4 sheet + resource document

Objective: draw a Bézier curve

Course of the activity:

1. Presentation of Bézier curves
2. Presentation of the Bézier curve construction algorithm
3. Construction of the Bézier curve by the students on A4 sheet (resource document)
4. Balance sheet

Student work:

On a white A4 sheet, place 3 points A, B, C (quite distant from each other) and construct the Bézier curve by iterating the algorithm presented.

#### **4. Documents used**

A resource document for Bézier Curves

Fiche-bezier.pdf (univ-bpclermont.fr)

#### **5. Webography**

Art nouveau à tous les étages (4) : l'apogée de la ligne courbe (decoatouslesetages.fr)  
courbes élémentaires (free.fr)

Petits bonshommes en graphisme - Les cahiers de Joséphine (canalblog.com)

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